When you don't know what to do…

Problem Solve!

Vermont High School Mathematics Portfolio Task Book

October 1995

Table of Contents

Introduction

Problem Solving

This book is based on a particular understanding of problem solving:

“Problem solving is what you do when you don’t know what to do!”

This means that students can only complete a **problem solving** task if two requirements are met:

1) The task itself must have the potential to be a **problem-solving task** (rather than an exercise). How do you know if the task has the potential to require problem-solving? You know this if it is capable of being approached in more than one way and does not lay out the approach to take.

2) In addition, the task must be a problem-solving task (rather than an exercise) **for your students**. How do you know if the task requires problem-solving for your students? A task requires problem-solving for students who don’t already know how to approach the task from past experience.

So . . . Use these tasks when they are still problems for your students. Let them solve the problems on their own or in groups. Ask them to document their work and their thinking. Use their final solutions as a transition to understanding new mathematical ideas.

The Tasks in This Book

The tasks found in this book were selected by attendees at the June, 1995 Research Scoring. These tasks have been used by Vermont teachers during the pilot years of the High school Mathematics Assessment. They were selected based upon their ability to:

- elicit a range of responses
- elicit the Vermont Criteria
- provide opportunities to solve problems in all the Vermont Content Clusters.

Some of the tasks are available in multiple versions, allowing you to customize the presentation for the particular needs of your students.

How This Book is Organized

The tasks in this book are organized in alphabetical order by title. Each task appears on a page suitable for photocopying for your class. Each task is followed by information about the task:

- a list of the primary content clusters covered by the task
- a list of the mathematics opportunities provided by the task
- a note telling how you might fit the task into your curriculum
- a solution including an answer and a possible approach to the problem

Primary Content Clusters

You may have noticed tthe Tables of Contents lists each task twice. First the tasks are listed in alphabetical order by title—the way they are arranged in this book. Second, they are organized in alphabetical order by its **Primary Vermont Content Cluster(s)**. Why?

Because integrating problem solving into an existing mathematics curriculum through portfolio assessment may seem to some like an “add on,” the second listing is designed to help teachers find tasks that relate to or extend their existing curriculum. In fact, the tasks in this book were chosen because they can easily be placed into the curriculum either as a pre-assessment, a post- assessment, or as a problem that helps construct mathematics. In order to help you see these opportunities, we have

- included the chart showing the Vermont Content Clusters on page 3,
- labeled each task with the Content Clusters it primarily fits with, and
- listed the tasks by Content Cluster in the Table of Contents.

Mathematics Opportunities in the Task: This section explains the possible mathematics opportunities that are in the task. Many of the opportunities are available for more than one task. After your class has completed a task, you may want to glance back at the opportunities and make instructional decisions for the future based on which opportunities students did (and didn't) take advantage of. Don't expect individual students to use all the mathematical opportunities available in a task.

Fitting it in: This section suggests some ways you might want to fit the task into your curriculum. It tells the minimum knowledge students need to be able to approach the task and how the task might fit into your curriculum. When a student solution is available in the Benchmarks, a note is included in this section.

It is also possible to adapt tasks for the needs of your students. You might, for example, present a diagram where none is given, to help students visualize the problem. If you do decide to make changes, be careful to consider how the changes affect the primary content cluster, the mathematical opportunities, the possible answers, the possible approaches, and the potential of the reworked task to elicit the Vermont criteria.

A Solution: This section provides you with an answer and possible approach to the problem posed. It is important to remember that all these tasks have many possible approaches, and some have more than one possible answer.

Student solutions should include an explanation of their understanding of the problem and their assumptions. Look carefully at student assumptions as you evaluate their approach and answer. Keep in mind that the solutions in this book are given for your information, but are not appropriate student responses because they don't address the criteria.

Vermont Content Clusters

The Vermont content clusters are condensed versions of the NCTM Standards content strands. Students should be provided with the opportunity to solve problems in all the content clusters.
The bullets under each content cluster are not all-inclusive but provide a framework in which to think about each content cluster.

Functions/Algebra

Tasks that provide the student with the opportunity to:

- Model real-world phenomena using a function.
- Represent and analyze relationships using tabular, symbolic, verbal, or graphical representations of functions.
- Examine or apply the general properties and behavior of classes of functions.
- Represent situations that involve numeric or variable quantities with expressions, equations, inequalities, or matrices.
- Operate on expressions or matrices.
- Solve algebraic problems using equations, formulas, inequalities, or matrices.
- Use tables or graphs as tools to interpret expressions, equations, or inequalities.

Geometry

Tasks that provide the student with the opportunity to:

- Represent problem situations with geometric models and deduce or apply the properties of figures such as perpendicularity, parallelism, congruence, and similarity.
- Analyze problem situations with coordinate representations of geometric figures and deduce or apply the properties of these figures using coordinates, vectors, or transformations.
- Apply the concepts of perimeter, area, volume, and angle measure.
- Show an understanding of an axiomatic system by investigating and comparing various geometries.

Probability and Statistics

Tasks that provide the student with the opportunity to:

- Derive and apply summary statistics such as measures of central tendency, variability, regression, and correlation from single or two-variable data.
- Construct and draw inferences from charts, tables, or graphs that summarize data from real-world situations.
- Use curve fitting.
- Use sampling, hypothesis testing, and experimental design techniques to examine real-world situations.
- Use experimental or theoretical probability models, as appropriate, to represent and solve problems involving uncertainty.
- Use simulations to estimate probabilities.

Trigonometry, Discrete Mathematics, Underpinnings of Calculus, and Mathematical Structure

Tasks that provide the student with the opportunity to:

- Apply trigonometry to problem situations involving triangles.
- Explore periodic real-world phenomena using sine and/or cosine functions.
- Determine maximum or minimum points of a graph and interpret results.
- Investigate limiting processes by examining infinite sequences or series.
- Examine and interpret the areas under a curve.
- Represent problem situations using discrete structures such as finite graphs, matrices, sequences, and recurrence relationships.
- Represent finite graphs using matrices.
- Compare and contrast the real number system and its subsystems with regard to structure character.
- Show an understanding of the logic of algebraic procedures.
- Use the underlying structures of mathematical systems in problem-solving situations.

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The tasks found in this book were selected by attendees at the June, 1995 Research Scoring. Thanks extend beyond this group to the many Vermont teachers who submitted portfolios during the pilot projects.

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Bagels or Donuts

Name ______________________________ **Date** ______________________________

Your school has offered you an opportunity to run a breakfast concession in the cafeteria. You will be allowed to sell either bagels or donuts, but not both. Your market research indicates that students are indiscriminate consumers of toroidal breakfast food: they would buy the same quantity of either product at the planned selling price of 50 cents each. Therefore, your choice of which item to sell will depend upon which you can obtain at a lower cost.

Your bagel supplier requires that you make a $200.00 initial deposit to cover the franchise fees and delivery costs for the rest of the school year. After that, you can buy as many bagels as desired at a cost of 12 cents per bagel. On the other hand, you can purchase donuts with no initial charge, at a cost of 30 cents each for the first 500 donuts, and 20 cents for each additional donut.

a) What is the smallest number of each item that you would need to sell to have a profitable business?

b) At what level of sales would a bagel or donut business be equally profitable?

c) If you were expecting a small volume of sales, would you rather sell bagels, or donuts, or neither?

Give evidence to support your findings.

Adapted from a Balanced Assessment Task
Balanced Assessment is funded by the National Science Foundation

Bagels or Donuts

Primary Content Clusters: Functions
Algebra

Mathematics Opportunities: This task provides the student with the opportunity to:
- build and use linear functions
- build and use a linear function that is defined differently for different parts of the domain
- compare and find the intersection of two linear functions
- understand the real costs of running a business
- use graphic or numeric organizational tools

Fitting it in: This problem is accessible to all students. It can be solved by making lists of outcomes and comparing lists. Algebraic solutions can be reached by students who have knowledge of linear equations or those who have learned about linear relationships. This task can be used prior to learning about linear relationships, and then used instructionally after the students have solved the problem to build the mathematical concept. Or, the task can be used as a post assessment. Anytime problem solving tasks are used as post assessments, it is best NOT to give the task right after the student has learned the skill or concept, but at a later time.

A Solution: The solution provided is only a SAMPLE.

An Answer:

a) Donuts are profitable from the first one sold. You must sell at least 527 bagels before a profit is reached.

b) You must sell 1,875 pieces of each product for an equal profit.

c) For a low sales volume selling donuts is a better choice because it is profitable.

One Approach:

a) The cost of bagels in dollars as a function of n bagels bought is:

$C_{bagels}(n) = 200 + .12n$

and the cost of donuts as a function of n donuts bought is:

$C_{donuts}(n) = .3n \qquad n \leq 500$
$C_{donuts}(n) = 150 + .2(n - 500) \qquad n > 500$

The profit function in dollars for bagels is:

$P_{bagels}(n) = .5n - C_{bagels}(n) = .5n - (200 + .12n) = .38n - 200$

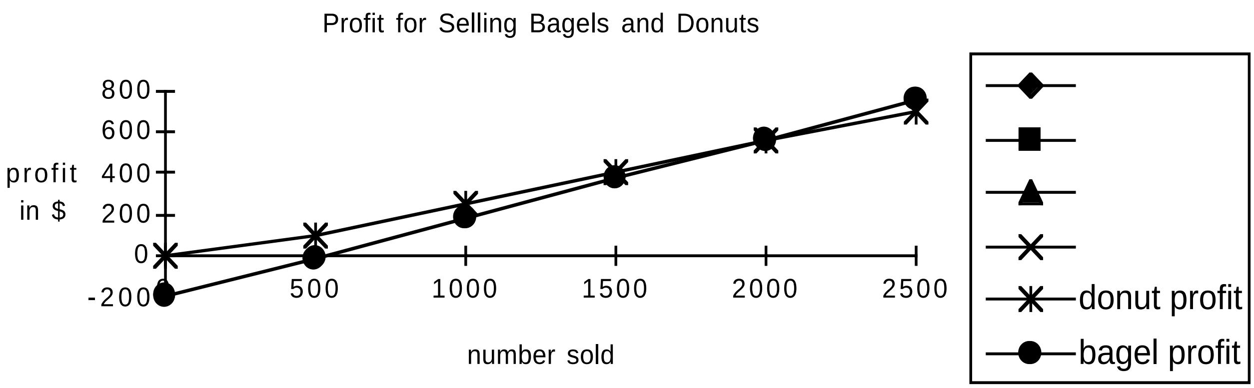
The profit function in dollars for donuts is:

$P_{donuts}(n) = .5n - C_{donuts}(n) = .5n - .3n = .2n \qquad n \leq 500$

$P_{donuts}(n) = .5n - C_{donuts}(n) = .5n - [150 + .2(n - 500)] = .3n - 50 \qquad n > 500$

When your profit function is negative, you lose money on a sale (costs exceed income).

The donut business is profitable from the very first sale, because the selling price exceeds the unit cost and there is no additional up front charge, whereas the bagel business is only profitable after overcoming the initial charge. To determine at which point the profit becomes positive, we can use the formula above and solve the inequality $0.38n - 200 > 0$ or we can graph the profit function and locate the n-axis intercept. Either method gives a solution of $n \approx 527$. Graphed, the profit function looks like this:



b) The number of pieces sold that make the bagel and donut businesses equally profitable may be obtained by equating the bagel profit function and the donut profit function.

$.3n - 50 = .38n - 200$

$n = 1,875$

An alternative approach is to find the point on the graph where the two profit functions intersect (1,875, 512.50). At a sales level of 1,875 items, bagels and donuts each yield a profit of $512.50.

c) For a low sales volume ($n < 527$) selling donuts is profitable, but selling bagels creates a loss.

FYI: There is a point at which selling bagels becomes more profitable than selling donuts. If you sell 2,000 bagels, your profit in dollars is:

$.38(2,000) - 200 = 560$

and if you sell 2,000 donuts, your profit in dollars is:

$.3(2,000) - 50 = 550$

Therefore, selling 2,000 bagels is $10 more profitable, about a 2% difference in profit between bagels and donuts.

The Bean Left in the Urn Problem

Name ______________________________ **Date** ______________________________

An urn contains 75 white beans and 150 black beans. Next to the urn is a large pile of black beans. Beans are removed and replaced according to certain rules.

THE RULES

- On each turn, two beans are removed at random. That is, a person reaches into the urn and, without looking, removes two beans.
- If at least one of the beans is black, place one black bean on the extra pile, and drop the other bean, black or white, back into the urn.
- If both beans are white, throw them away, take one bean from the extra pile (which are all black) and drop it back into the urn.

On each turn, two beans are removed and one bean is added; therefore, the number of beans in the urn decreases by one. At some point there will be only one bean left.

What color will it be??

Contributed by Ken McCaffrey, Brattleboro Union High School, Brattleboro, Vermont

The Bean Left in the Urn Problem

Primary Content Cluster: Discrete Mathematics

Mathematics Opportunities: This task provides the student with the opportunity to:
- do a problem which is most easily done if they start at the end
- use logic and organization
- articulate an explanation which requires both numerical and logical support
- use modeling to solve a problem

Fitting it in: This problem is within reach of all students and can be used at any time in the curriculum.

A Solution: The solution provided is only a SAMPLE.

An Answer:

The final bean must be white.

One Approach:

Each WW pair reduces the total white by 2, while each pair including a black (either WB or BB) leaves the total white the same.

i.e.	WW	$\rightarrow$Place 1B in urn	73 W left
	WB	$\rightarrow$Replace 1W in urn	75 W left
	BB	$\rightarrow$Replace 1B in urn	75 W left

Starting with 75 white beans, each time a pair is removed there will be an <u>odd</u> number of white beans left. According to the rules, the last white bean (1W left) can never be removed.

WB	$\rightarrow$Replace 1W in urn	1 W left
BB	$\rightarrow$Replace 1B in urn	1 W left

So regardless of the order in which pairs are drawn, the final bean must be white.

(Students should note that initial condition of an odd # of white beans is important in this argument.)

Boarding up a Window

Name ______________________________ **Date** ______________________________

A man had a square window on the south side of his cabin. In the winter, too much light came through it, so he decided to board up part of it. Always looking for a stylish solution, he settled on covering part of the window by making a smaller square out of wood, the corners of the square of wood just touched the midpoints of the sides of the original window. What fraction of the original window is the “new” window?

Not one to rest on his laurels, he decided to make a similar change to a regular hexagonal window in his bathroom. In other words, he made a smaller hexagon out of wood; the vertices of the hexagon of wood just touched the midpoints of the sides of the original window. Again, what fraction of the original window is the new window?

Boarding up a Window

Primary Content Cluster: Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- apply specialized area formulas
- apply the Pythagorean theorem
- use properties of 30-60-90 triangles
- develop sophisticated and specialized area formulas

Fitting it in: This task can be introduced after students have studied the properties and areas of 45-45-90 and 30-60-90 triangles, but could be introduced to those who only have backgrounds which include finding the area of triangle, and the Pythagorean theorem. Solutions will vary depending on which of these tools students have at their disposal.

A Solution: The solution provided is only a SAMPLE.

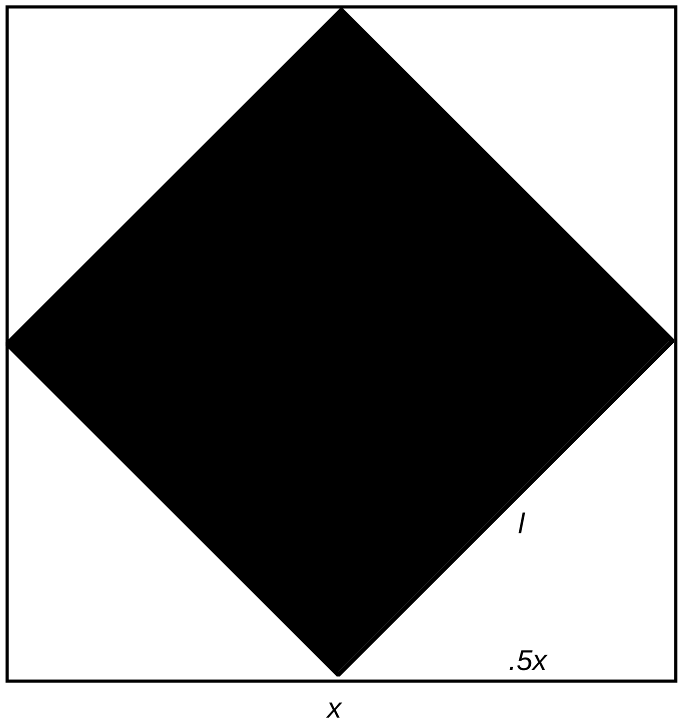
An Answer:

For the square window: $\frac{new\ area}{old\ area} = \frac{1}{2}$

For the hexagonal window: $\frac{new\ area}{old\ area} = \frac{1}{4}$

One Approach:

The square problem was interpreted as pictured below.



Area of original square $=x^2$

The length (*l*) of the side of the new square can be found using the Pythagorean theorem:

$l = \sqrt{(.5x)^2+(.5x)^2} = x\sqrt{.5}$ or $.5x\sqrt{2}$

Therefore, the area of the board added is:

$(x\sqrt{.5})^2 = .5x^2$

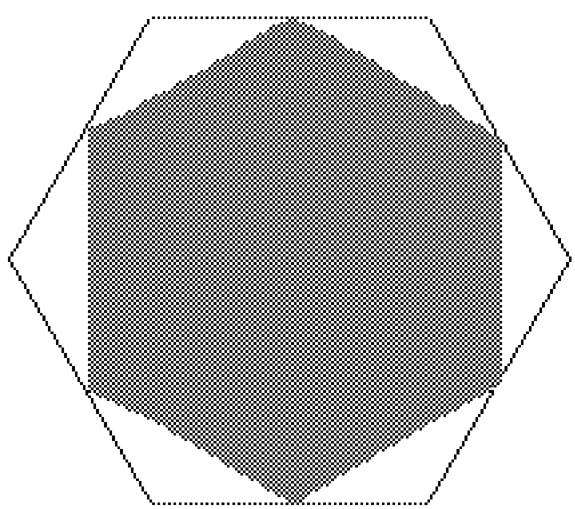
And subtracting this from the area of the original square gives the area of the new window as:

$.5x^2$

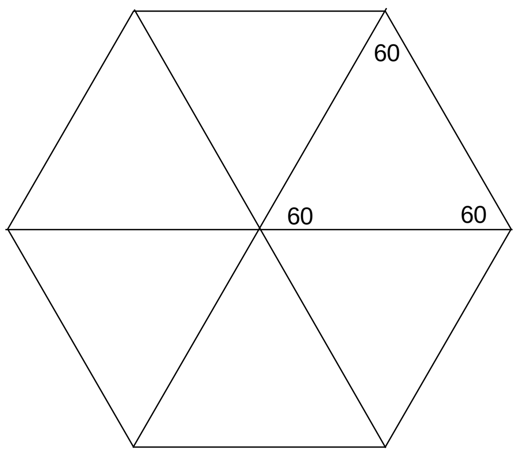
$$\frac{area\ of\ new\ window}{area\ of\ old\ window} = \frac{.5x^2}{x^2} = \frac{1}{2}$$

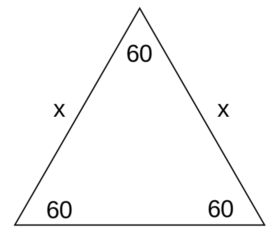
The area of the new window is half the area of the original window.

For the hexagonal case the problem was interpreted as pictured below:



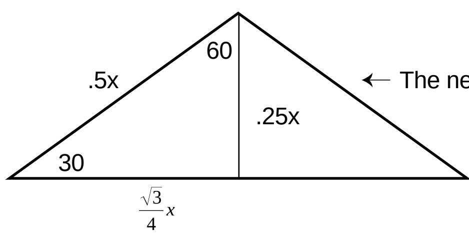
The original hexagonal window can be viewed as being composed of 6 equilateral triangles:





The area of one equilateral triangle is: $\frac{\sqrt{3}}{4}x^2$ where x is one side.

So the area (A) of the original hexagonal window is: $A = 6(\frac{\sqrt{3}}{4}x^2) = \frac{3\sqrt{3}}{2}x^2$



The area of one of the twelve 30-60-90 triangles is: $\frac{1}{2}bh = \frac{1}{2}(\frac{\sqrt{3}}{4}x)(.25x)$

So the area (A) of the new window is: $A = 12(\frac{1}{2})(\frac{\sqrt{3}}{4}x)(.25x) = \frac{3\sqrt{3}}{8}x^2$

The area of the new window is one quarter of the original hexagonal window.

$$\frac{\textit{area of new window}}{\textit{area of old window}} = \frac{\frac{3\sqrt{3}}{8}x^2}{\frac{3\sqrt{3}}{2}x^2} = \frac{1}{4}$$

Books from Andonov

Name ______________________ **Date** ______________________

While on a vacation in Andonov, an American tourist couple decided to send home various books they had bought there to include in their private library. When entering the post office in a small Andonovian town, they saw the following chart of postal rates on a wall:

Weight in tallos	Price in ords
Under 5	1
5 or more, but less than 10	4
10 or more, but less than 15	9
15 or more, but less than 20	16
20 or more, but less than 25	25
25 or more, but less than 30	36
30 or more, but less than 35	49

The couple was confused by the sign. When they asked the clerk if packages weighing 35 tallos or more were allowed, the clerk indicated there was a 100 tallos weight limit on each package. However, he offered no help to determine the cost of boxes that are heavier than those listed on the sign. The couple was on their own.

Part 1: The couple has eleven books: 5 of them weigh 12 tallos each; 3 of them weigh 2.5 tallos each; 2 of them weigh 6 tallos each; and 1 of them weighs 3.5 tallos. How would you recommend they package the books in order to minimize the shipping costs?

Part 2: **After determining how to ship their books, the couple decided that the post office needed a new sign.** This sign would contain a set of instructions that a person could use to determine the cost of shipping any package weighing up to 100 tallos. Determine a set of instructions and then explain them fully.

Adapted from a Balanced Assessment Task
Balanced Assessment is funded by the National Science Foundation

Books from Andonov

Primary Content Cluster: Functions

Mathematics Opportunities: This task provides the student with the opportunity to:
- recognize patterns in given data
- extend patterns
- express a function in his/her own words
- generalize a pattern

Fitting it in: All students can use this problem. It can be used effectively at any time in the curriculum.

A Solution: The solution provided is only a SAMPLE.

An Answer:

53 ords

One Approach:

Part 1:

Notes:

• As weight increases arithmetically, price increases in a "geometric" pattern.

• The best deal is at the highest end of the lowest interval.

• Each book is at its minimum shipping cost unless two or more books can be combined such that the total weight of the two remains in the interval of the heavier.

• The only book combinations that make sense (because they increase the weight of a package without increasing shipping cost) are these:

12 tallos and 2.5 tallos
6 tallos and 2.5 tallos

6 tallos and 3.5 tallos

WEIGHT	ORDS:
12 + 2.5 = 14.5	9
12 + 2.5 = 14.5	9
12 + 2.5 = 14.5	9
12	9
12	9
6 + 3.5 = 9.5	4
6	4
	53

53 ords is the minimum cost. In effect, you pay only for shipping the seven heavy books.

Part 2:

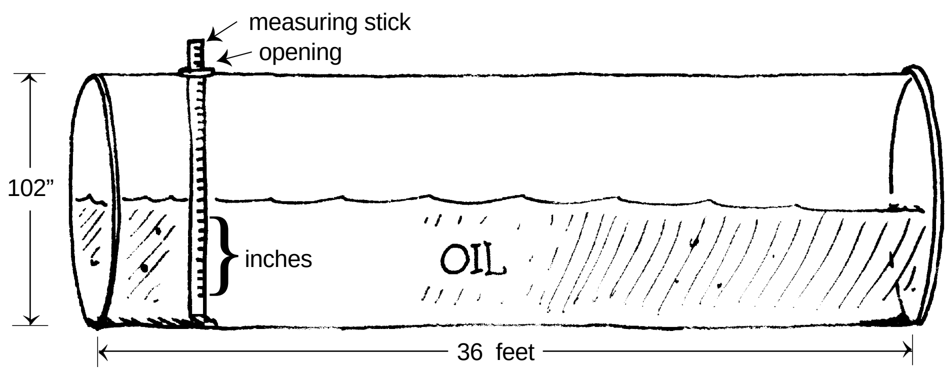
<u>One Set of Instructions:</u>

Take the weight of a package and divide it by 5. Disregarding any remainder, add one to the quotient. Square the result. This final number will be the price in ords for mailing the package.

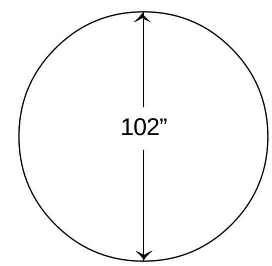
The B U H S Oil Tank Problem

Name ______________________________ **Date** ______________________________

At Brattleboro Union High School, the heating oil is stored in a right cylindrical tank represented by the following diagram:



Side View



Front View

The school needs a chart giving the number of gallons of oil in the tank as a function of the number of inches indicated by the measuring stick.

Make such a chart: Use the units digit of the number of the day of the month on which you were born, as inches, as the first “inches” measurement in your chart. Go by ten-inch increments after that.

For example: If you were born on April 23, you would start with 3 inches. Figure out the number of gallons in the tank when the stick indicates the oil is 3 inches deep in the tank, 13 inches, 23 inches, etc., up to 93 inches. Notice that the tank is 102 inches high.

This task was contributed by Ken McCaffrey, Brattleboro Union High School, Brattleboro, Vermont

The B U H S Oil Tank Problem

Primary Content Clusters: Geometry
Trigonometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- work with a three-dimensional figure
- use trigonometry
- use geometry
- work with a cylinder

Fitting it in: This problem is appropriate after right-triangle theory at the end of geometry, or during the study of trigonometry.

A Solution: The solution provided is only a SAMPLE.

An Answer:

See chart on page 21.

One Approach:

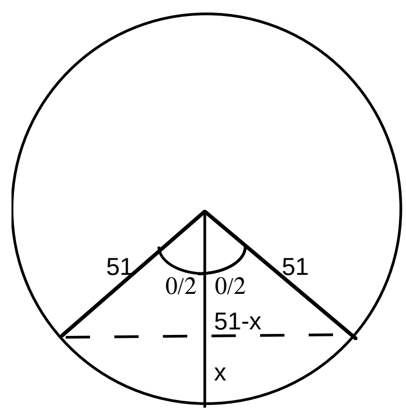
Find the inch intervals that you need to use for your chart. Draw a a cross-section of the tank, and draw a horizontal line across the diagram to represent the level of oil. Let x equal the depth of oil in the tank in inches. Determine the cross-sectional area of oil using the following formulas:

I. When $0 < x \leq 51$:

Area of segment = area of sector - area of triangle

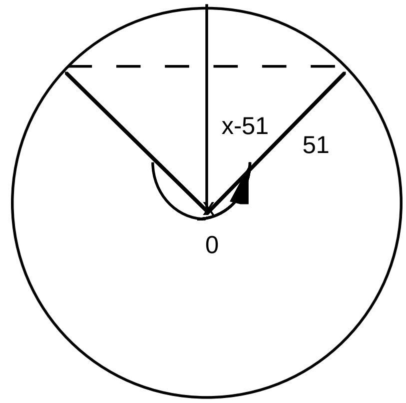
$$= \frac{\theta}{360}\pi r^2 - \frac{1}{2}bh$$

$$= \frac{2\cos^{-1}\left(\frac{51-x}{51}\right)}{360}\pi(51)^2 - \frac{1}{2}\left[2\sqrt{51^2-(51-x)^2}\right](51-x)$$



For example: when $x = 3$:

$$A = \frac{2\cos^{-1}\left(\frac{48}{51}\right)}{360}\pi(51)^2 - \left(\sqrt{51^2 - 48^2}\right)(48) \approx 69.35\text{in.}^2 \approx .4816\text{ft.}^2$$



Volume = (cross-sectional area)(length) $\approx .4816 \cdot 36 \approx 17.3377\text{ft.}^3$

Using the conversion factor $1\text{ft.}^3 \approx 7.481\text{gal.}$

Volume $\approx 130\text{gal.}$

II. When $51 \le x \le 102$:

Cross-sectional Area = area of sector + area of triangle

$$= \frac{\vartheta}{360}\pi r^2 + \frac{1}{2}bh$$

$$= \left[\frac{360 - 2\cos^{-1}\left(\frac{x-51}{51}\right)}{360}\right]\pi(51)^2 + \frac{1}{2}\left[2\sqrt{51^2 - (x-51)^2}\right](x-51)$$

For example, when $x = 63$

$$A = \left[\frac{360 - 2\cos^{-1}\left(\frac{12}{51}\right)}{360}\right]\pi(51)^2 + \left(\sqrt{51^2 - 12^2}\right)(12) \approx 5{,}298.25\text{in.}^2 \approx 36.793\text{ft}^2$$

Volume = (cross-sectional area)(length) $\approx 36.793 \cdot 36 \approx 1{,}324.56\text{ft.}^3$

Using the conversion factor $1\text{ft.}^3 \approx 7.481\text{gal.}$

Volume $\approx 9{,}909\text{gal.}$

NOTE: Answers may vary slightly from those in the chart, due to differences in rounding.

Inches	Gallons	Inches	Gallons	Inches	Gallons
1	25	35	4639	69	11001
2	71	36	4821	70	11179
3	130	37	5004	71	11355
4	199	38	5188	72	11530
5	277	39	5373	73	11703
6	364	40	5559	74	11874
7	457	41	5745	75	12043
8	556	42	5933	76	12211
9	662	43	6121	77	12376
10	772	44	6309	78	12539
11	888	45	6499	79	12699
12	1009	46	6688	80	12858
13	1134	47	6878	81	13013
14	1263	48	7069	82	13166
15	1397	49	7259	83	13316
16	1534	50	7450	84	13463
17	1674	51	7641	85	13607
18	1818	52	7831	86	13747
19	1965	53	8022	87	13884
20	2115	54	8212	88	14018
21	2268	55	8403	89	14147
22	2423	56	8593	90	14272
23	2582	57	8782	91	14393
24	2742	58	8972	92	14509
25	2905	59	9160	93	14619
26	3070	60	9348	94	14725
27	3238	61	9536	95	14824
28	3407	62	9722	96	14917
29	3578	63	9908	97	15004
30	3751	64	10093	98	15082
31	3926	65	10277	99	15151
32	4102	66	10460	100	15210
33	4280	67	10642	101	15256
34	4459	68	10822	102	Full– 15282

Buried Treasure #1

Name ______________________________ **Date** ______________________________

You are on a trip to one of the Cartesian Islands. In the cellar of your cabin you have found an old map that claims to hold the secret location of a buried treasure. The map shows four trees at locations A, B, C, and D. The trees at A and D are along an east-west road that runs through the island. The tree at B is directly north of the tree at A, and the tree at C is directly north of the tree at D.

The map's directions say to trace a path from the tree at B to the tree at D, and to do the same for the trees at A and C. The treasure is buried halfway between the intersection of these paths (point E) and the road. Because three small lakes dot the island, you are only able to measure the distances between B and D, A and C, and A and D:

B to D 300 paces
A to C 260 paces
A to D 240 paces

To find the location of the treasure, you must determine the distance from E to the road and then walk half that distance from E. How far is E from the road and how far from E is the buried treasure?

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National Council of Teachers of Mathematics 9-12 Geometry Addenda

(Note to Teacher: There are two versions of this problem. #1 has no diagram. #2 provides students with a diagram. Choose the one that is most appropriate for your students.)

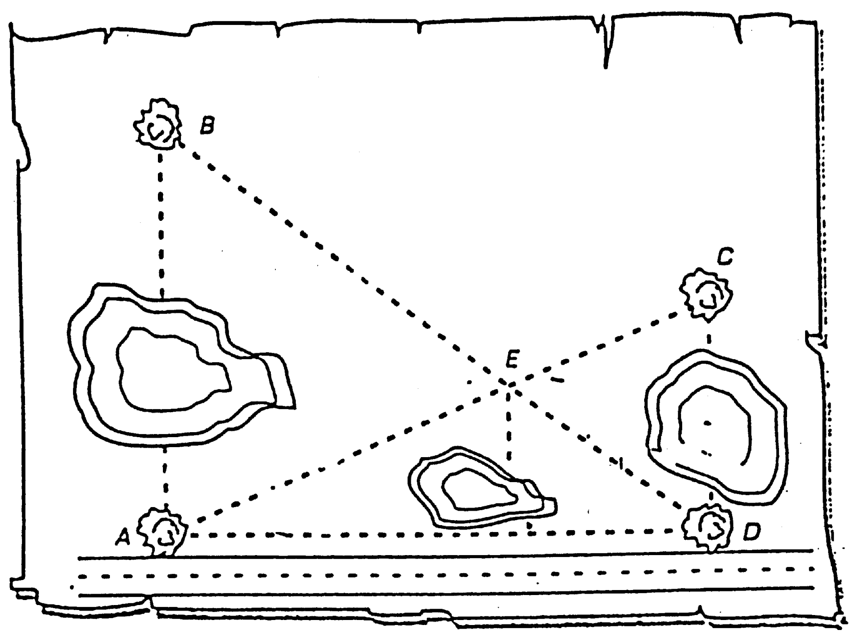
Buried Treasure #2

You are on a trip to one of the Cartesian Islands. In the cellar of your cabin you have found an old map that claims to hold the secret location of a buried treasure. The map shows four trees at locations A, B, C, and D. The trees at A and D are along an east-west road that runs through the island. The tree at B is directly north of the tree at A, and the tree at C is directly north of the tree at D.

The map's directions say to trace a path from the tree at B to the tree at D, and to do the same for the trees at A and C. The treasure is buried halfway between the intersection of these paths (point E) and the road. Because three small lakes dot the island, you are only able to measure the distances between B and D, A and C, and A and D:

B to D 300 paces
A to C 260 paces
A to D 240 paces

To find the location of the treasure, you must determine the distance from E to the road and then walk half that distance from E. How far is E from the road and how far from E is the buried treasure?



Buried Treasure #1 and #2

Primary Content Clusters: Algebra
Geometry
Trigonometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- sketch a map from distances and compass directions
- apply the Pythagorean theorem
- use properties of similar triangles
- use trigonometric functions
- model the solution
- use scale to solve a problem

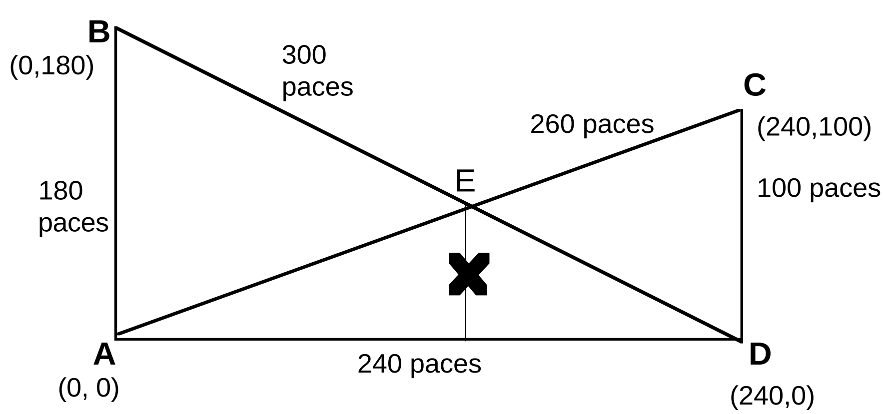
Fitting it in: This task is accessible to all students. Students have solved this problem by: modeling on a football field; drawing scale models; using a) the Pythagorean theorem, b) similarity of triangles, and c) right-triangle trigonometry. Timing of the use of this task depends upon your goal. If you want to assess if your students will use the Pythagorean theorem or similar triangles in a problem solving context, then use it well after the study of these topics. If you want to use the task as a starting point for studying these topics, then have students solve the problem first, and use it as model of the application of the mathematics as they learn it. For a student solution, refer to the Benchmarks.

A Solution: The solution provided is only a SAMPLE.

An Answer:

E is approximately 64 paces from the road and 32 paces from the buried treasure.

One Approach:



Using the Pythagorean theorem: $\overline{CD} = 100$

$\overline{AB} = 180$

So, the equation of the line of which AC is a segment is: $y=\frac{100}{240}x=\frac{5}{12}x$

And, the equation of the line of which BD is a segment is: $y=-\frac{180}{240}x+180=-\frac{3}{4}x+180$

Solving the two equations simultaneously to find the intersection yields:

$$\frac{5}{12}x=-\frac{3}{4}x+180$$
$$5x=-9x+2160$$
$$14x=2160$$
$$x\approx 154.3$$
$$y\approx 64.3$$

Since the answer is in paces, it makes sense to round to:

$$x\approx 154$$
$$y\approx 64$$

Point E is approximately (154,64) and the Treasure is half the distance from E to line segment AD. So, the coordinates of the Treasure are approximately (154,32).

E is approximately 64 paces from the road.
E is approximately 32 paces from the buried treasure.

Can of Soup

Name ______________________________ **Date** ______________________________

On the desk is a can of soup. The can is a closed circular cylinder that requires a certain amount of metal. Your task is to find what dimensions you could use for a new soup can that would hold the same amount of soup but would require the least amount of metal.

Note that the new soup can should still be a closed circular cylinder. You can ignore any overlapping metal at the seams.

This task was contributed by Bob Peek, Champlain Valley Union High School, Hinesburg, Vermont

Can of Soup

Primary Content Clusters: Geometry
Underpinnings of Calculus

Mathematics Opportunities: This task provides the student with the opportunity to:
- make critical assumptions that affect the solution (size of the original can)
- work with cylinders
- use formulas for volume and surface area of the cylinder
- use techniques for finding and classifying extrema
- compare actual and theoretical dimensions

Fitting it in: This task is most applicable after students have studied applications of first and second derivatives for determining maxima and minima. However, it can be solved numerically by systematic trial and error by any student who has or can develop the formulas for volume and surface area of the cylinder.

A Solution: The solution provided is only a SAMPLE.

An Answer: New radius: 3.69 cm
New height: 7.37 cm

One Approach: The dimensions of the original can were agreed to by the class to be:
Height: 9.5 cm
Diameter: 6.5 cm

Using standard volume and surface area formulas:

$V = \pi r^2 h$ and $A = 2(\pi r^2) + Lh$ where $r = radius;\ l = height;\ L = circumference\ of\ the\ circular\ ends$

The volume of the original can is $100.34\pi\ cc.$ So, to keep that volume constant,

the new can must have dimensions where $h = \dfrac{100.34}{r^2}$.

Find the radius, x, needed to minimize the surface area:

$$y = 2\pi x^2 + 2\pi x(100.34x^{-2})$$
$$y = 2\pi x^2 + 200.68\pi x^{-1}$$
$$y' = 4\pi x - 200.68\pi x^{-2}$$

Setting $y' = 0$ and solving yields $x = 3.69\ cm$ as the radius of the new can.

Then, $h = 7.37\ cm$.

In general, the height and the diameter (2x) should be the same.

The problem can also be solved by numerical methods using a spreadsheet or a calculator.

Consecutive Addends

Name ______________________________ **Date** ________________________

You come into your classroom and see these sums on the board......

15= 7 + 8

7 = 3 + 4

14 = 2 + 3+ 4 + 5

20 = 2 + 3+ 4+ 5 + 6

12 = 3 + 4+ 5

6 = 1 + 2 + 3

83 = 41 + 42

21 = 6 + 7 + 8

15 = 4 + 5 + 6

You notice that each of the sums is the sum of consecutive addends. Your teacher suggests that the sums on the board provide some interesting questions and patterns to explore.

Your task is....

- to identify patterns in these sums,
- make some conjectures about the patterns or other observations you make,
- and then investigate your observations, patterns. and questions.
- **Finally,** based upon your investigations state conclusions supported with data from your investigation about the sum of any number of consecutive addends.

This task was adapted from a Balanced Assessment Task developed at
Michigan State University and the University of Nottingham.

BA902

Consecutive Addends

Primary Content Clusters: Algebra
Discrete Mathematics

Mathematics Opportunities: This task provides the student with the opportunity to:
- pose problems/questions
- systematically investigate conjectures
- find patterns, describe/generalize patterns, and give supporting evidence to show why the patterns/rules are correct

Fitting it in: This is a good introductory problem for all students learning to make mathematical generalizations. It can be used at any time in the curriculum.

A Solution: The solution provided is only a SAMPLE.

An Answer:

Many solutions are possible. As you assess student work, look for their justifications for the conclusions they draw, and test their generalizations. Assess the conjectures that students make, not the conjectures you think they should have made.

One Approach:

Some students may look for a way to predict the sum of any number of consecutive addends. They might make a chart like the following, in which x represents the highest number in the sequence, and n represents the number of addends:

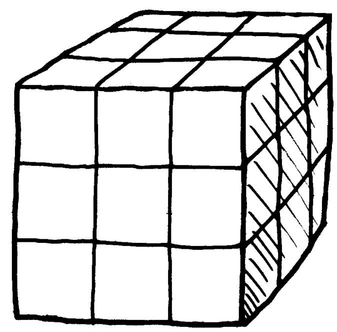
Sum of the Sequence	Number of Consecutive Addends
$2x$-1	2
$3x$-3	3
$4x$-6	4
$5x$-10	5
$nx - \frac{[n(n-1)]}{2}$	n

NOTE: Students may not even look for this generalization. That is okay. Assess the conjectures that students make, not the conjectures you think they should have made.

Cubes

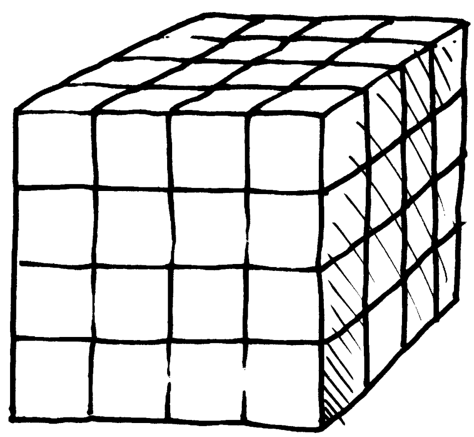
Name ____________________ **Date** ____________________

This 3 x 3 x 3 cube is made out of 27 small cubes.



In the drawing, 19 small cubes can be seen. The other 8 small cubes are hidden.

1. Now look at this 4 x 4 x 4 cube:



   (a) How many small cubes can be seen?
   (b) How many are hidden?

2. In a drawing of a 5 x 5 x 5 cube:

   (a) How many small cubes can be seen?
   (b) How many are hidden?

3. Explain how you could find the number of small cubes that are visible and hidden in a cube of any size.

This task was adapted from a Balanced Assessment Task developed at
Michigan State University and the University of Nottingham.

BA912

Cubes

Primary Content Clusters: Algebra
Geometry
Discrete Mathematics

Mathematics Opportunities: This task provides the student with the opportunity to:
- generate and recognize a number pattern
- express a number pattern in their own words
- generalize a pattern
- determine the volume of a cube
- distinguish between volume and surface area
- work with a figure that is partly concealed

Fitting it in: This is a good introductory problem for all students learning to make mathematical generalizations. It can be used at any time in the curriculum.

A Solution: The solution provided is only a SAMPLE.

An Answer:

1. a) 37 visible
   b) 27 hidden

2. a) 61 visible
   b) 64 hidden

3. Many acceptable explanations can be produced to determine the number of cubes visible and the number of cubes hidden in a cube of any size. It is important to establish if a given explanation is sufficiently general to refer to 'any size' cube.

One such explanation is: Take a cube of any size, say, n^3. Then the number of small cubes that are hidden is $(n-1)^3$. Using this, the number of small cubes that can be seen is given by $n^3-(n-1)^3=(3n^2-3n+1)$.

Students may arrive at other generalizations of this problem depending upon the approach they take.

One Approach:

In this case, a generalization was made based on volume formulas. This generalization was then used to determine the answers to all three parts.

Diagonals in a Polygon

Name ______________________________ **Date** ______________________________

How many diagonals does a 30-sided polygon have?

Diagonals in a Polygon

Primary Content Clusters: Algebra
Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- make critical assumptions that affect the solution (whether the polygon is convex)
- organize data in a number of different ways
- generate and recognize patterns
- generalize patterns

Fitting it in: This problem is appropriate for all students who have an understanding of polygons and diagonals. It is a good task for students to solve in the study of geometry before they begin studying theorems relating to polygons. For a student solution, refer to the Benchmarks.

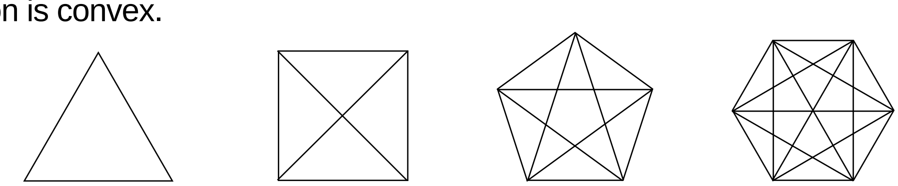
A Solution: The solution provided is only a SAMPLE.

An Answer:

A 30-sided polygon has 405 diagonals.

One Approach:

Assume the polygon is convex.



0 diagonals 2 diagonals 5 diagonals 9 diagonals

Number of Sides	Number of Diagonals	
3	0	
4	2	$1+1$
5	5	$2+2+1$
6	9	$3+3+2+1$
7	14	$4+4+3+2+1$
...	...	...
n	$n-3+\frac{(n-3)(n-2)}{2}$	$n-3+\sum_{i=1}^{n-3} i$

The number of diagonals simplifies to: $\frac{n^2-3n}{2}$

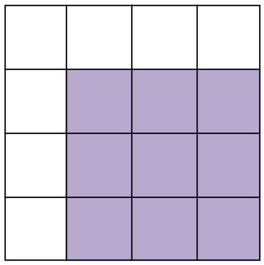
So, for $n=30$, $\frac{900-90}{2}=405$

Therefore a 30-sided polygon has 405 diagonals.

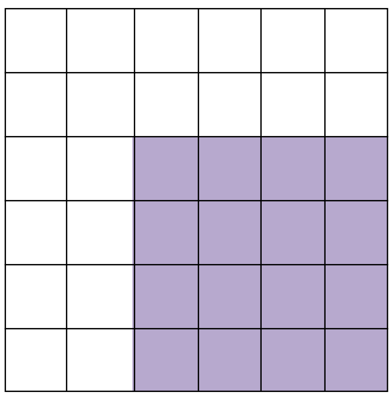
The Difference of Two Squares

Name ______________________ **Date** ______________________

Both 7 and 20 may be written as the difference of two squared whole numbers.



$$4^2 - 3^2 = (4 \cdot 4) - (3 \cdot 3)$$
$$= 16 - 9$$
$$= 7$$



$$6^2 - 4^2 = (6 \cdot 6) - (4 \cdot 4)$$
$$= 36 - 16$$
$$= 20$$

Which other whole numbers may be written as the difference of two squares?

This task was adapted from a Balanced Assessment Task
Balanced Assessment is funded by the National Science Foundation

The Difference of Two Squares

Primary Content Cluster: Algebra

Mathematics Opportunities: This task provides the student with the opportunity to:
- learn elementary number theory
- develop organized tables
- apply binomial expansion
- generate and recognize a pattern
- generalize a pattern

Fitting it in: This is a good problem for students who understand the difference of squares and binomial expansion and who are learning to make mathematical generalizations. Other students can do this problem by collecting data that shows patterns over a number of cases, but they may not reach a generalization in algebraic terminology.

A Solution: The solution provided is only a SAMPLE.

An Answer: All odd numbers and all multiples of 4.

One Approach: The differences are found in the double-lined box below:

Square Numbers	**0**	**1**	**4**	**9**	**16**	**25**	**...**
0	0	1	4	9	16	25	...
1	-1	0	3	8	15	24	...
4	-4	-3	0	5	12	21	....
9	-9	-8	-5	0	7	16	...
16	-16	-15	-12	-7	0	9	...
25	-25	-24	-21	-16	-9	0	...
...	...	...	...	...	...	...	...

There are several patterns evident in this table.
Reading down the main diagonal we obtain 0, 0, 0, 0....
Moving one cell right each time we obtain 1, 3, 5, 7, 9,.... (odd numbers)
Moving one cell right again, we obtain 4, 8, 12, 16,.... (multiples of 4) ...and so on.

It is evident from the table that all odd numbers and all multiples of four can be expressed as the difference of two squared whole numbers.

The only whole numbers that *cannot* be expressed as the difference of two squared whole numbers lie in the sequence 2, 6, 10, 14, 18. This sequence can be generalized as $4n-2$ where $n \geq 1$.

Distances in a Room

Name ____ **Date** _________________________________

Consider a large empty room. Its dimensions are 6 meters long, 5 meters wide, and 3 meters high. You need to run a phone wire from one corner at the floor to the opposite corner at the ceiling. Find the shortest length of wire which will accomplish the task.

Adapted by Rufus Winsor, Thetford Academy, Thetford, Vermont; and
Don Robinson, Green Mountain Union High School, Chester, Vermont

Distances in a Room

Primary Content Cluster: Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- work with three dimensional figures
- make critical assumptions that affect the solution (the need for a practical solution)
- use the Pythagorean theorem
- consider the Pythagorean theorem in three dimensions

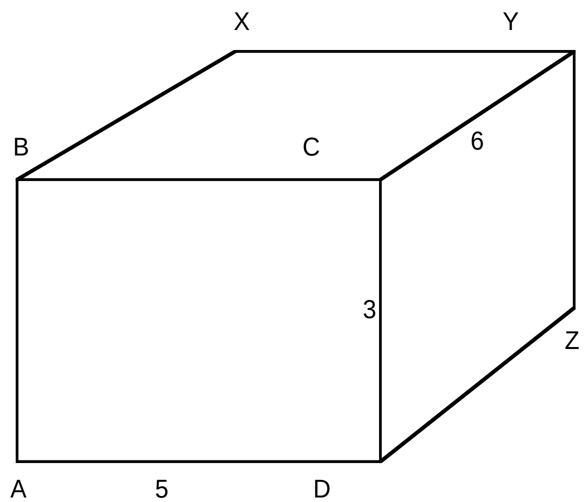
Fitting it in: This problem is excellent for students who have just studied the Pythagorean theorem, but can be used at any time thereafter.

A Solution: The solution provided is only a SAMPLE.

An Answer:

The shortest distance is 8.37 m. The most practical solution is 10 m.

One Approach:

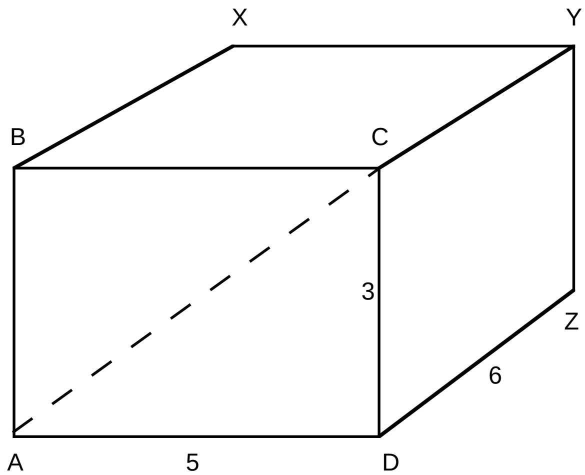


1. Many routes can be found that use the edges of the figure, each route including one length, one width, and one height, for example:

$\overline{AY} = \overline{AD} + \overline{DC} + \overline{CY}$

All these routes will be 14m:

$5 + 3 + 6 = 14$



2. Another approach uses one diagonal and one edge, for example:

$\overline{AY} = \overline{AC} + \overline{CY}$

$\overline{AC} = \sqrt{5^2 + 3^2} = \sqrt{25 + 9} = \sqrt{34} \approx 5.83$

$\therefore \overline{AY} \approx 5.83 + 6 = 11.83m$

Similarly:

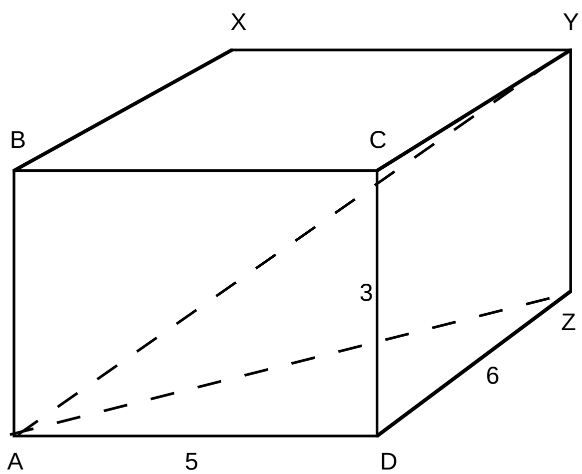
$\overline{AY} = \overline{AD} + \overline{DY} =$

$5 + \sqrt{3^2 + 6^2} = 5 + \sqrt{45} \approx 5 + 6.71 \approx 11.71m$

and

$\overline{AY} = \overline{AB} + \overline{BY} =$

$3 + \sqrt{5^2 + 6^2} = 3 + \sqrt{61} \approx 3 + 7.81 \approx 10.81m$



3. Using a direct route between A and Y shortens the distance by several meters. But it isn't a practical solution because the wire would be an obstacle in the room.

$\overline{AY} = \sqrt{(\overline{YZ})^2 + (\overline{AZ})^2}$ and $\overline{AZ} = \sqrt{(\overline{AD})^2 + (\overline{DZ})^2}$

$\overline{AY} = \sqrt{(\overline{YZ})^2 + (\sqrt{(\overline{AD})^2 + (\overline{DZ})^2})^2}$

$\overline{AY} = \sqrt{(\overline{YZ})^2 + (\overline{AD})^2 + (\overline{DZ})^2}$

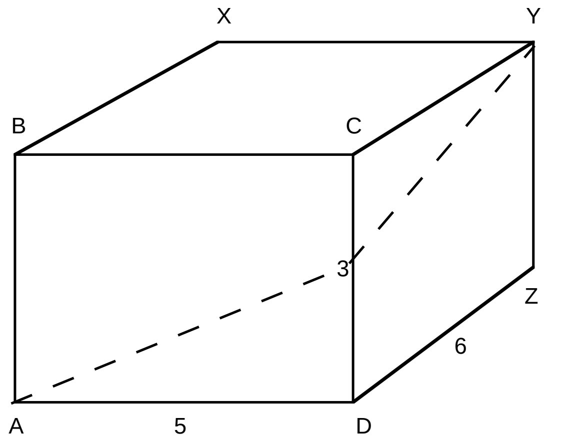
height width length

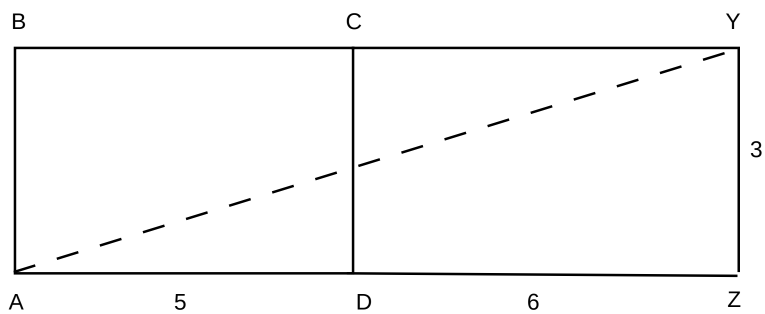
$\overline{AY} = \sqrt{3^2 + 5^2 + 6^2} \approx 8.37m$

4. The best practical solution is to put the wire either through the walls:

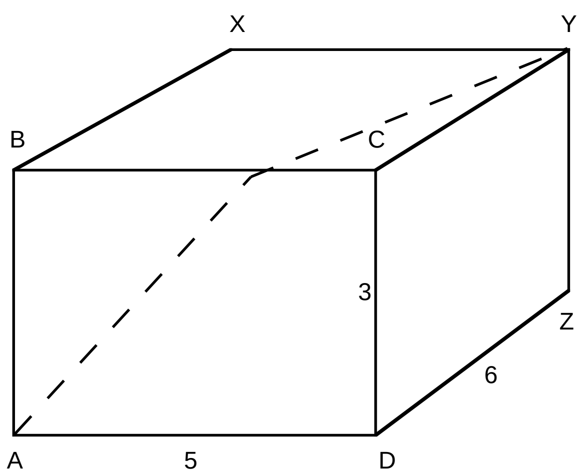
$\overline{AY} = \sqrt{11^2 + 3^2} =$

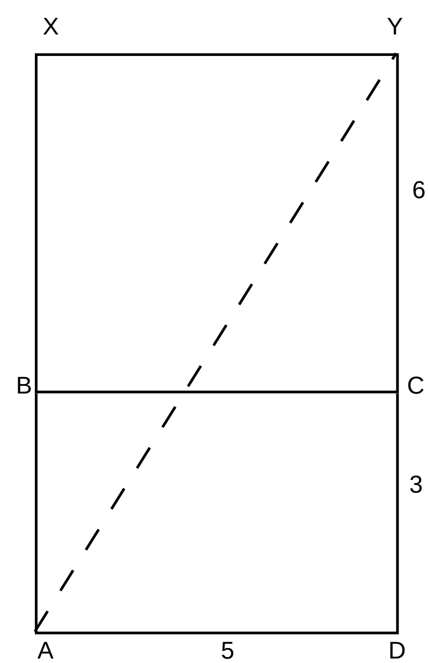
$\sqrt{121 + 9} = \sqrt{130} \approx 11.4m$





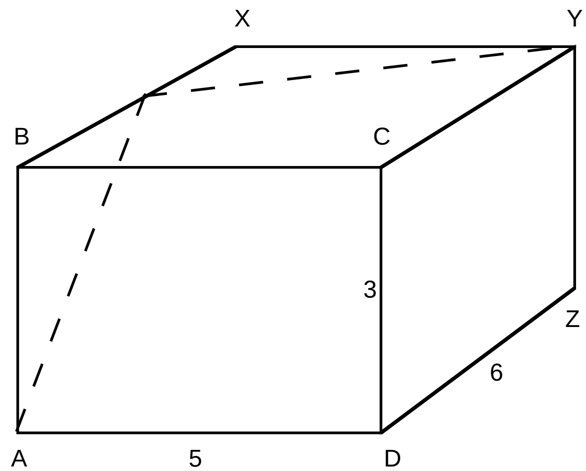
or, better yet, through the wall and ceiling:





$\overline{AY} = \sqrt{9^2+5^2} = \sqrt{81+25} = \sqrt{106} \approx 10.3m$

Using the OTHER wall and ceiling (i.e., running the wire from A across the segment $\overline{BX}$ and from there to Y):



$\overline{AY} = \sqrt{8^2 + 6^2} = \sqrt{64+36} = \sqrt{100} = 10$m

The best practical solution is 10m

E-I-E-I-O # 1

Name ______________________ **Date** ______________________

Mr. Rainville had a farm, E-I-E-I-O.
And on this farm he built a fence, E-I-E-I-O.
He wants a rectangle here.
With 120 meters of wire there.

What dimensions should Mr. Rainville use to have the maximum possible fenced-in area? E-I-E-I-O.

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

E-I-E-I-O #2

Name ______________________ **Date** ______________________

Mr. Rainville had a farm, E-I-E-I-O.
And on this farm he built a fence, E-I-E-I-O.
He wants a rectangle here.
With 120 meters of wire there.

Part 1:

What dimensions should Mr. Rainville use to have the maximum possible fenced-in area? E-I-E-I-O.

Part 2:

How does the problem change and what would be the dimensions of the yard if he still used 120 meters of wire but he used the side of the barn for one of the sides of the rectangular yard. Remember: he still wants to have the maximum area possible using only 120 meters of fence.

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

E-I-E-I-O Versions 1 and 2

Primary Content Clusters: Algebra
Geometry
Underpinnings of Calculus

Mathematics Opportunities: This task provides the student with the opportunity to:
- build a function
- find and compare results
- make critical assumptions that affect the solution in Version 2 (dimensions of the barn)
- apply completing the square
- analyze a quadratic equation (parabola)
- apply first and second derivative tests for extrema

Fitting it in: There are two versions of this task. The first asks the student to only deal with the wire. The second asks the student to deal with the wire using the side of a barn. The choice of which version to use can be based on students' abilities and where you are in the curriculum. This task is accessible to all students using a simple chart listing different dimensions. It can be introduced when the notion of function is developed, used when students have studied analysis of the parabola, or used with calculus students who are learning to apply tests of extrema.

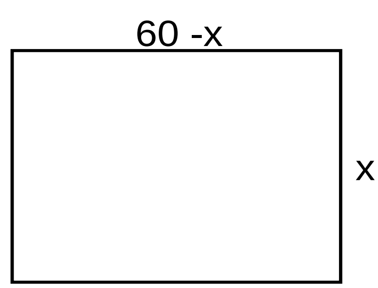
A Solution: The solution provided is only a SAMPLE.

An Answer:

The farmer should create a square yard, 30 meters by 30 meters.

One Approach:

Version One and Part One of Version Two:



$$A(x) = x(60 - x) = 60x - x^2$$

$$0 < x < 60$$

By **algebra:**

$$A(x) = -x^2 + 60x$$
$$= -(x^2 - 60x)$$
$$= -(x^2 - 60x + 900) + 900$$
$$= -(x - 30)^2 + 900$$

If this parabola is graphed, the vertex is (30,900) and the parabola opens down. The maximum is when

$x = 30$ Area $= 900m^2$

Or, by **calculus**:

$$A(x) = -x^2 + 60x$$
$$A'(x) = -2x + 60$$
$$-2x + 60 = 0$$
$$x = 30$$

Check to see if this value is a maximum value:

$$A''(x) = -2$$

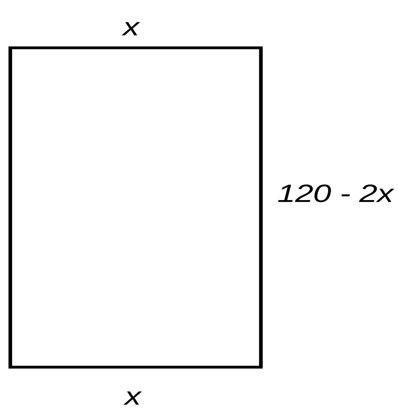
Maximum area of $900m^2$ occurs when the rectangle is a square, with sides $x = 30m$.

Part Two of Version Two:

An Answer: Assuming that the barn is at least 60 meters long, the farmer can use 60 meters of the side of the barn to build a 30 meter by 60 meter yard, doubling his area to 1,800 square meters.

One Approach: The equation for area in this case is : $Area = x(120 - 2x)$
Students may use algebra or calculus or a spreadsheet (see below).

By **calculus:**



$$y = x(120 - 2x)$$
$$= -2x^2 + 120x$$
$$y' = -4x + 120$$
$$0 = -4x + 120$$
$$x = 30$$

$y'' = 4$ extrema will be max.

Or, by using a **spreadsheet:**

The spreadsheet shown on the following page explores the areas of various yards made with a barn side from 28 meters long to 118 meters long. The barn side forms one length of the rectangle and the 120 meters of fencing forming two widths and a length.

Spreadsheet

width	length	area	
1	118	118	
2	116	232	
3	114	342	
4	112	448	
5	110	550	
6	108	648	
7	106	742	
8	104	832	
9	102	918	
10	100	1000	
11	98	1078	
12	96	1152	
13	94	1222	
14	92	1288	
15	90	1350	
16	88	1408	
17	86	1462	
18	84	1512	
19	82	1558	
20	80	1600	
21	78	1638	
22	76	1672	
23	74	1702	
24	72	1728	
25	70	1750	
26	68	1768	
27	66	1782	
28	64	1792	
29	62	1798	
30	60	1800	INTEGRAL SOLUTION
31	58	1798	
32	56	1792	
33	54	1782	
34	52	1768	
35	50	1750	
36	48	1728	
37	46	1702	
38	44	1672	
39	42	1638	
40	40	1600	
41	38	1558	
42	36	1512	
43	34	1462	
44	32	1408	
45	30	1350	
46	28	1288	

Holy Cow #1

Name ______________________________ **Date** ______________________________

A cow is tethered to a ring on the outside wall of a barn, 10 meters from one of the corners. The barn is 20 meters wide by 40 meters long. The rope is 50 meters in length. Find the cow's grazing area.

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

Holy Cow #1

Primary Content Clusters: Geometry
Underpinnings of Calculus

Mathematics Opportunities: This task provides the student with the opportunity to:
- determine area of semi- and quarter circles
- make critical assumptions that affect the solution (fencing or equipment in the barn yard; location, manner of opening, and status of the barn doors; side of the barn to which the cow is tied—short or long side?)

Fitting it in: This task is suitable for all students. It is best used well after the introduction of the formula for the area of a circle.

Solutions: The solutions provided are only a SAMPLE.

An Answer:

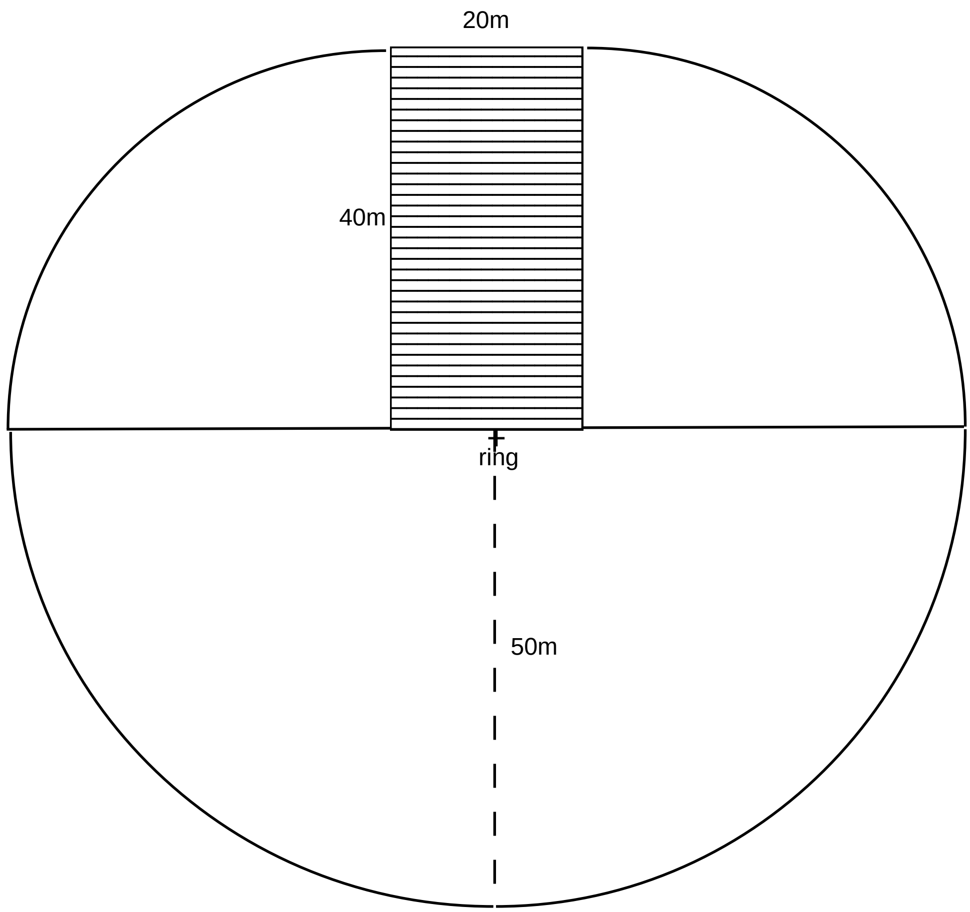
There are two solutions to this problem.

1) If the rope is place ten feet from the corner on the short side of the barn, the area is approximately 6,437 square meters.

2) If the rope is placed ten meters from a corner on the long side of the barn, the area is approximately 5,809 square meters.

These solutions are not based upon other factors the students bring into the problem, such as an open barn door obstructing the cow's path, or a fence around the barn. Therefore, check the execution of the task based upon all the factors they are considering.

One Approach:



The cow is tethered on the short side of the barn. The area of the grazing range is:

$$A = \frac{1}{4}\pi(40)^2 + \frac{1}{4}\pi(40)^2 + \frac{1}{2}\pi(50)^2$$

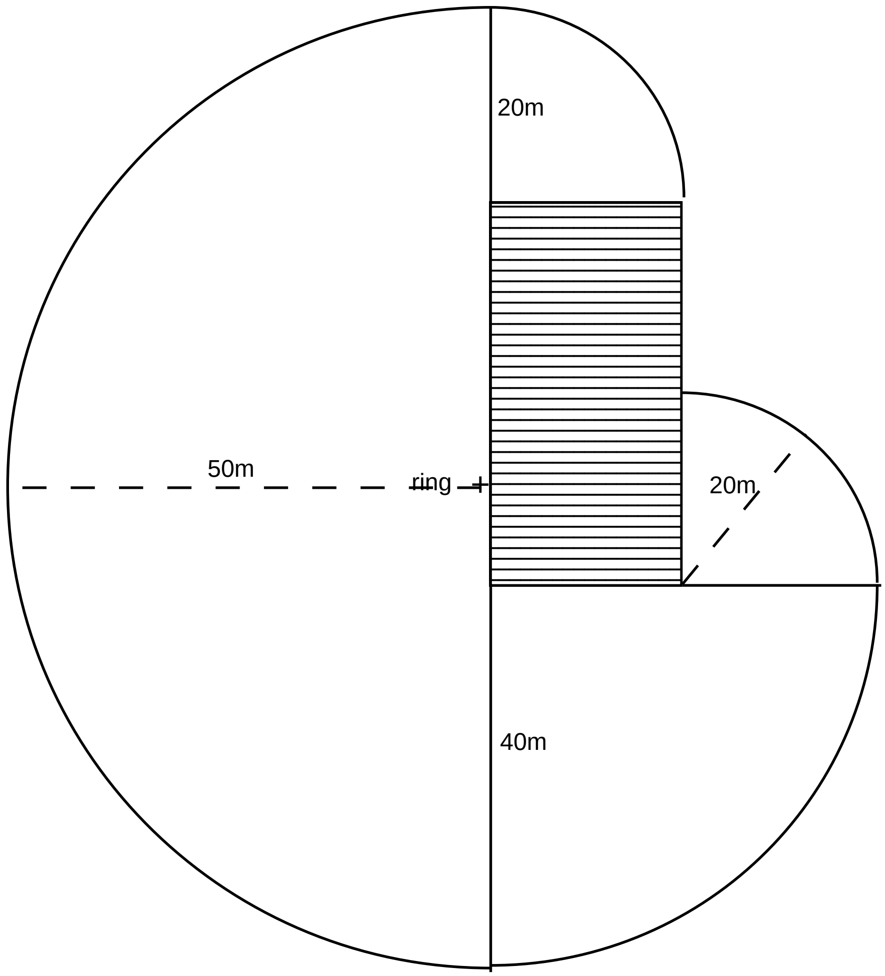
$$A = \frac{1}{2}\pi(40)^2 + \frac{1}{2}\pi(50)^2$$

$$A \approx (\frac{1}{2}\cdot 3.14\cdot 1,600) + (\frac{1}{2}\cdot 3.14\cdot 2,500)$$

$$A \approx 2,512 + 3,925$$

$$A \approx 6,437m^2$$

But suppose the cow is tethered on the long side of the barn. Then the grazing range is:



$$A = \frac{1}{4}\pi(20)^2 + \frac{1}{4}\pi(20)^2 + \frac{1}{4}\pi(40)^2 + \frac{1}{2}\pi(50)^2$$

$$A = \frac{1}{2}\pi(20)^2 + \frac{1}{4}\pi(40)^2 + \frac{1}{2}\pi(50)^2$$

$$A \approx (\frac{1}{2} \cdot 3.14 \cdot 400) + (\frac{1}{4} \cdot 3.14 \cdot 1,600) + (\frac{1}{2} \cdot 3.14 \cdot 2,500)$$

$$A \approx 628 + 1,256 + 3,925$$

$$A \approx 5,809\text{m}^2$$

How Much Gold Can You Carry Out? #1

Name ______________________ **Date** ______________________

A vault contains a large amount of gold, and you are told that you may keep as much gold as you can carry out under the following conditions:

- On the first trip you may only take one pound.
- On each successive trip you may take half of the amount you carried out on the previous trip.
- You may make one trip every minute.

What is the total amount of gold that you will be able to carry out and how long will it take to do it?

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

How Much Gold Can You Carry Out? #2

Name ______________________ **Date** ______________________

A vault contains a large amount of gold, and you are told that you may keep asmuch gold as you can carry out under the following conditions:

- On the first trip you may only take one pound
- On each successive trip you may take half of the amount you carried out on the previous trip.
- You may make one trip every minute.

Using the value of gold at $350 per ounce, determine your <u>hourly</u> rate of earnings if you worked for:

a) only the first fifteen minutes
b) only the first twenty minutes
c) one full hour

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

How Much Gold Can You Carry Out? #1 and #2

Primary Content Cluster: Underpinnings of Calculus

Mathematics Opportunities: This task provides the student with the opportunity to:

- develop organized tables
- work with exponential functions
- work with relative rates
- generate and recognize patterns
- generalize a pattern

Fitting it in: This is a good problem for all students learning to make mathematical generalizations. Students will approach it in different ways depending on the level of sophistication of their backgrounds. This problem can be used when teaching about exponential functions. It is an equally good problem to use to introduce sequences and series.

Version 1:

A Solution: The solution provided is only a SAMPLE.

An Answer:

You will never be able to reach 2 lb. of gold.

One Approach:

Minutes (Trips)	Pounds	Total
1	1	1
2	.5	1.5
3	.25	1.75
4	.125	1.875
5	.0625	1.9375

You can take $\frac{1}{2^{n-1}}$ for the nth trip when $n =$ minutes

You can take almost, but never quite, 2 lb.

Version 2:

Students will use various approaches to find the total amount removed after the passage of n minutes
Some will note the pattern: $2 - \frac{1}{2^{n-1}}$

The problem asks for *hourly* wage. Note units:

$$n = 15$$

$$\text{Wages} = \frac{(2 - \frac{1}{2^{14}})\text{lb.} \cdot (\frac{16\ \text{oz.}}{\text{lb.}}) \cdot (\frac{\$350}{\text{oz.}})}{15\ \text{min.}\left(\frac{1\ \text{hr.}}{60\ \text{min.}}\right)}$$

$$\text{Wages} = \frac{\$11{,}199.66}{\frac{1}{4}\text{hr.}} = \$44{,}798.63 \text{ per hour}$$

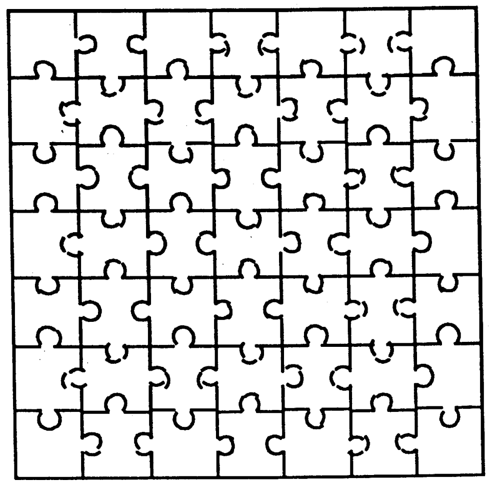
Similarly,

$n = 20$ Wages $= \$33{,}599.97$ per hour

$n = 60$ Wages $= \$11{,}200.00$ per hour

Jigsaw Puzzle Two

Name ______________________________ **Date** ______________________________

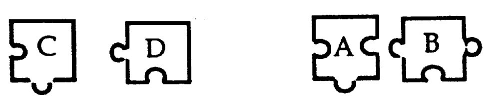


Imagine that all the puzzle pieces are colored blue on just the front side. In this task, you are not allowed to flip pieces over so that the blue side is on the table.

1. A square jigsaw puzzle has 625 pieces. It is made with pieces that have one straight edge, two straight edges, or no straight edges. How many of each of these types of pieces are used? Describe how you obtained your solution.

2. How many of each type would you need for a square puzzle of any size?

3. In the puzzle there are two different types of pieces with only one straight edge, A and B. There are also two different types of corner pieces, C and D.



- Explain clearly why you always need an equal number of A and B pieces for rectangular jigsaw puzzles of any size, **regardless** of what corner pieces you use.

4. Another square jigsaw puzzle has 2,500 pieces. How many of **each type** of the pieces **(A, B, C, D, and center pieces)** do you need to make a puzzle of 2,500 pieces?

This task was adapted from a Balanced Assessment Task developed at the University of Nottingham, UK.

BA903

Jigsaw Puzzle Two

Primary Content Clusters: Algebra
Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- recognize patterns in data
- extend patterns
- analyze spatial relationships
- generalize patterns

Fitting it in: This task is appropriate for all students who have the necessary skills to be able to generalize the pattern. It can be approached algebraically or by modeling. Although there are no geometric formulas required for this task, students can practice spatial sense and rotation.

A Solution:

An Answer:

1.

4	pieces with two straight edges
92	pieces with one straight edge
529	pieces with no straight edges
625	pieces altogether

2. $n =$ number of pieces on one side of puzzle

4	pieces with two straight edges
$4(n-2)$	pieces with one straight edge
$(n-2)^2$	pieces with no straight edges
$4+4(n-2)+(n-2)^2=n^2$	pieces altogether

3. Assume that puzzle pieces are based on squares with sides that have a length of one unit.
- The corner and the center pieces have an area of exactly one square unit.
- One edge piece has an area just **greater** (by one knob) than one square unit.
- The other edge piece has an area one knob **less than** one square unit.

Since the jigsaw puzzle has an area which is a whole number, there must be an equal number of each type of edge pieces.

4. Be careful if you use this question. The original writer of this task is concerned that students will miss the subtlety of this question. The solution is different than the solution to questions 2 and 3 because only one type of corner piece can be used to make a puzzle with an even number of pieces on a side.

4	pieces of one type of corner (all C **or** all D)
$\left[\frac{4(n-2)}{2}\right] = 96 + 96$	pieces with one straight edge (half A, half B)
$(n-2)^2 = 2,304$	center pieces
$4 + 4(n-2) + (n-2)^2 = n^2$	pieces altogether
2,500	pieces altogether

One Approach:

1., 2., and 4. Begin by figuring out the make-up of the 7 by 7 jigsaw puzzle pictured:

	4	pieces with *two* straight edges (corners)
	20	pieces with *one* straight edge
	25	pieces with *no* straight edges
TOTAL:	49	pieces altogether

With this information, the generalization for Question 2 can be worked out. Students may do this before they work out the answer for Question 1:

You always need 4 corners for a puzzle.
Number of corner pieces $= 4$

The number of pieces in the whole puzzle is the square of the number of pieces on each side.
Total number of pieces $= n^2$

Two of the pieces on any given side are corner pieces, the rest are pieces with one straight edge. Each side has one quarter of the straight pieces used in the puzzle.
Number of straight edge pieces $= 4(n-2)$

The rest of the pieces are center pieces.
Number of center pieces = $n^2 - 4(n-2) - 4 =$

$n^2 - 4n + 4 =$

$(n-2)^2$

Students can then use the formula to find the answer for Questions 1.

3. The reasoning used for Question 3 is explained above.

4. For Question 4, the following approach is possible:

A 2,500 piece puzzle has sides with 50 pieces per side, or 48 edge pieces and 2 corners per side. Any edge made up of an even number of pieces will be asymmetrical on its ends; i.e., if one end has a knob sticking out, the other end will have a space that needs a knob. The same shape corner piece (rotated 90° each time) will be needed on all 4 corners in order to complete the interlocking pattern.

Jump Ball Probability

Name ______________________________ **Date** ______________________________

One way to handle a "jump ball" situation when we play basketball without a referee would be to determine possession of the ball as follows:

- The two players face each other with a clenched fist and on the count of two each extends one, two, or three fingers.
- One player is selected to call "odd" or "even" as they both extend their fingers.
- The total number of extended fingers are counted and if the count matches the odd or even call, the caller gets the ball. Otherwise, the other team gets it.
- The caller will alternate between teams each time there is a new "jump ball" situation.

You need to answer two questions.

1) Is the odd/even method fair?

2) Is there any way that you can improve the odds in your favor when you are involved in a "jump ball"?

Contributed by Ray Henderson, Missisquoi Valley Union High School, Swanton, Vermont

Jump Ball Probability

Primary Content Cluster: Probability

Mathematics Opportunities: This task provides the student with the opportunity to:

- apply classical probability
- apply conditional probability
- use tree diagrams or other organizational tools
- anticipate and investigate changes in the game
- generate and recognize a pattern
- make critical assumptions that affect the solution (define a "fair" game)

Fitting it in: This problem is appropriate for all students
.
A Solution: The solution provided is only a SAMPLE .

This solution is written as an extended model that includes all the thinking behind the decisions made.

I am being asked two questions in this problem.

1. Is the game fair?

I am going to assume that in a fair game there is an equal probability that I win or lose. That is to say that the probability that I win will equal 1/2. There are many areas to look at in evaluating whether a situation is fair. This exercise may contain valuable lessons that can be applied in other circumstances.

2. Can I discover a strategy that will make the odds come out in my favor?

(This would also be a valuable lesson to apply to other areas.)

The rules of the game are as follows:

On the count of two each of two players will extend one, two, or three fingers. One player will call "odd" or "even." The extended fingers will be counted, and if the count matches the call, the caller wins, if not the opponent gets the ball. The teams will take turns calling the count.

I need to assume that both players will release their fingers at the same time the call is made so that no player will have the advantage of knowing how many fingers the opponent is extending and the caller must call before the count is known.

The game seems fair but I will make a sample space of the possible outcomes before I make my final decision. Here's how I determined the sample space.

1	1 - 1,1 even 2 - 1,2 odd 3 - 1,3 even	**2**	1 - 2,1 odd 2 - 2,2 even 3 - 2,3 odd	**3**	1 - 3,1 even 2 - 3,2 odd 3 - 3,3 even

There are 9 possible outcomes of which 5 are even and 4 are odd. I am starting to wonder if the game is fair because we have an odd number of outcomes and the number of odd and even events are not equal.

This leads me to an important question: Does it make a difference how many fingers I put out? If I put out one finger or three fingers the probability is 2/3 that the event will be even. If I put out two fingers the probability is also 2/3 that the event will be odd.

I think that I see a strategy that might work! When I get to call, I can improve my odds using the following combinations:

When I extend two fingers I call "odd", otherwise, I call "even." I can see from the tree diagram that the probability that I win is 2/3. I am assuming that my opponents do not know this trick and I must vary my call so they are not able to predict what I will do.

Now I need to consider if I can improve the odds if it is my opponent's turn to make the call? Now all I can control is the number of fingers that I put out.

First consider the case where I extend one finger and my opponent does not know the trick, that is to say, that the call, and the number of fingers that are extended by my opponent will be random.

Given that I extend <u>one finger</u>, what is the possibility that I win? The outcome will depend upon what my opponent calls and the count. There are six possible outcomes, given the probability of the call is 1/2 either way and the probability that any given finger is put out is 1/3.

E = Even call 0 = Odd call 1,2,3 = opponent's fingers
<u>P (EN 1) Means</u> Probability opponent called even and extended 1 finger

I extend 1 finger	P (EN 1) = (1/2) (1/3) = 1/6 I lose!	P (ON 1) = (1/2) (1/3) = 1/6 <u>I win!</u>
	P (EN 2) = (1/2) (1/3) = 1/6 <u>I win!</u>	P (ON 2) = (1/2) (1/3) = 1/6 I lose!
	P (EN 3) = (1/2) (1/3) = 1/6 I lose!	P (ON 3) = (1/2) (1/3) = 1/6 <u>I win!</u>

I win on three and my opponent wins on three, so the odds are equal that I will win if I extend one finger and everything else is equally likely.

The same logic can be applied to cases if I extend two or three fingers. So the game does appear to be fair if we assume that everything is random (e.g., if no one knows the trick, or if one person knows that trick but does not control the call).

I have answered both questions. I have found a way to improve my odds. I have also discovered that this is not a fair game if I get to call and I know the trick and my opponent does not. On the other hand, it is fair if neither person knows that trick or if the person who is calling does not know the trick.

Now I should find a way to confirm my results. I will try the empirical method with some friends and see if I can improve my odds. I will try fifty calls for each of us and record my results on a table. See the stem and leaf graphs attached to this report.

The empirical approach confirms my theoretical methods. It appears that the game is fair if everything is random. I know that I need more than 50 samples to get a better sample space but again there are real life time constraints here.

My stem and leaf graph shows that there were 26 out of 50 wins when the person did not know the trick, this is very close to a theoretical probability of 1/2. However, if I control the call and the number of fingers that I extend and my opponent does not know "my trick" the probability was 32/50 in my favor. This also was very close to the theoretical probability of 2/3. I should tell you that when we were making the stem and leaf graphs human error occurred several times and the caller who knew the trick made the wrong call, when this occurred we corrected the call to make our model correct. In the heat of the game this type of error might well occur more often. The human factor always has to be considered.

When I said that I was starting to wonder if the game was fair after I looked at the tree diagram I did not consider the fact that with the nine possible outcomes, *go two possible calls*, odd or even, so the sample space is doubled and then there are just as many winners as there are losers. So when everything is random the probability is 1/2.

Now let me consider an extension. Given the fact that we alternate calls and the best that I can do is control the call for half the time let me see if I can find the theoretical probability of winning in the long run. My opponent makes the calls 1/2 of the time, I will win 1/2 of my opponents calls for 1/4 of the total time. But when I make the call, which is 1/2 of the time I will win 2/3 of that time which is 1/3 of the total time. See below.

When opponent makes the calls:

the probability that I win is (1/2) x (1/2) = 1/4

When I make the call:

the probability that I win is (1/2) x (2/3) = 1/3

Since the two events are mutually exclusive we need to add the two probabilities.

$$1/4 + 1/3 = 7/12$$

If only one person knows that trick then the probability of that person winning is 7/12 and this would not be considered a fair game.

It would be interesting to consider if this a fair game when both sides know the trick.

Stem and Leaf Graph for "Jump Ball Probability"

1. **Stem** equals the number of caller's fingers
2. **Leaf** equals the number of opponent's fingers
3. A **circle** around number shows that the caller wins

* With Trick		Without Trick (Random)
① ① ① ① ① ① ① ① ① ① ① ①	1	1 1 1 1
2 2 2 2 2	1	2 ② ② 2 2 ② ② ② 2 ② ② ② ②
③ ③	1	③ ③ ③ 3 3 3
① ① ① ① ① ① ①	2	1 1 ①
2 2 2 2	2	② 2 2 2 2
3 3 3	2	3 3
① ① ① ① ①	3	① ① 1 ① ① ①
2 2 2 2 2 2 2 2 2	3	2 ② 2 ② 2 ② 2 2
③ ③ ③ ③ ③	3	③ ③ ③

caller wins $\frac{32}{50}$ caller wins $\frac{26}{50}$

Caller's Fingers

Data show that if caller knows trick: If the call is even, extend 1 or 3 finger(s)
If the call is odd, extend 2 fingers
Then, the probability of winning = 32/50

* On two occasions the caller who knew the trick made the wrong call.
Numbers were adjusted for his human error.

Mowing the Lawn

Name ______________________________ **Date** ______________________________

Chris and Lee are each responsible for mowing half of their 50-foot by 90-foot rectangular back yard. Chris starts mowing at an outside corner, gradually working toward the middle by mowing around the outside edges. If the mower cuts a 3-foot wide path, during which trip around the yard should Chris stop and Lee start mowing?

Adapted from a task submitted by Wendy Ovtasky,
Whitcomb Junior/Senior High School, Bethel, Vermont

Mowing the Lawn

Primary Content Cluster: Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- make critical assumptions that affect the solution (exactly how Chris mows)
- apply area formulas

Fitting it in: This task is accessible to all high school students. It can be used to pre-assess or to reinforce the application of determining the area of a rectangle in a problem-solving context.

A Solution: The solution provided is only a SAMPLE.

An Answer:

Lee should take over early in the fourth trip.

One Approach:

The area of the lawn $= 4{,}500\text{ft.}^2$

After Chris completes three trips, $2{,}196\text{ft.}^2$ will have been cut.

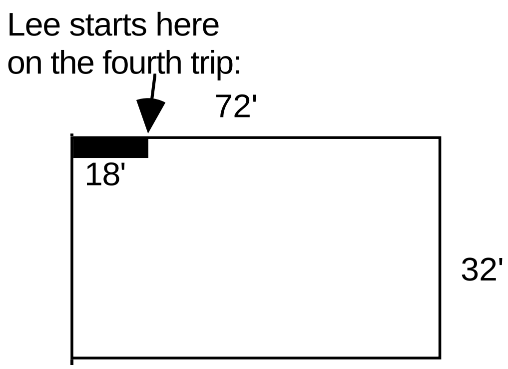
$$4{,}500 - (72)(32) = 2{,}196\text{ft.}^2$$

After four trips, $2{,}784\text{ft.}^2$ will have been cut.

$$4{,}500 - (66)(26) = 2{,}784\text{ft.}^2$$

So, Lee should take over sometime during the fourth trip.

Students might calculate the exact spot:



Half of the Lawn = 2,250 square feet. After three trips, 2,196 square feet are cut, so Chris only needs to mow another 54 square feet before turning the mower over to Lee. Mowing an additional eighteen-foot-long swath will take care of this, since the swath will be three feet wide and 3(18) = 54.

NEPTUNE vs. MARS

Name ____ Date ________________________________

A NEPTUNE car costs $12,500 and gets 40 miles per gallon of gasoline. A MARS car costs $11,000 and gets 31 miles per gallon. Gas costs $1.24 per gallon.

Nancy buys a NEPTUNE, and Mel buys a MARS.

a. Describe the cost of buying a NEPTUNE and driving it for x miles. Describe the cost of buying a MARS and driving it for x miles.

b. How many miles would Nancy and Mel need to drive before the costs for their cars would be the same?

c. How many miles would Nancy and Mel need to drive for the difference in their costs to be $500?

Contributed by Rufus Winsor, Thetford Academy, Thetford, Vermont

NEPTUNE vs. MARS

Primary Content Clusters: Functions
Algebra

Mathematics Opportunities: This task provides the student with the opportunity to:
- build and use linear functions
- compare and find the intersection of two linear functions
- understand the real costs of running a vehicle
- to use graphic or numeric organizational tools
- solve simultaneous equations

Fitting it in: The context of this problem makes it accessible to all students. It can be solved by making lists of outcomes and comparing lists. Algebraic solutions can be reached by students who have knowledge of linear functions. This task can be used prior to learning about linear relationships, and then used instructionally after the students have solved the problem to build the mathematical concept. Or, the task can be used as a post assessment. Anytime problem solving tasks are used as post assessments, it is best NOT to give the task right after the student has learned the skill or concept, but at a later time.

A Solution: The solution provided is only a SAMPLE.

An Answer:

a. The cost in dollars of buying and driving a NEPTUNE x miles is:
$12,500 + .031x$

The cost in dollars of buying and driving a MARS x miles is:
$11,000 + .04x$

b. They will have to drive approximately 166,667 miles before their costs are the same.

c. Students may describe the difference in costs in words or by equations. They will have to drive approximately 111,111 miles for the difference in costs to be \$500.

One Approach:

a. Total cost of NEPTUNE $= 12,500 + 1.24\left(\frac{x}{40}\right) = 12,500 + .031x$

Total cost of MARS $= 11,000 + 1.24\left(\frac{x}{31}\right) = 11,000 + .04x$

b. Solving simultaneous equations:

$$12,500 + 1.24\left(\frac{x}{40}\right) = 11,000 + 1.24\left(\frac{x}{31}\right)$$

$x \approx 166,667$ miles

c. $\left[12,500 + 1.24\left(\frac{x}{40}\right)\right] - \left[11,000 + 1.24\left(\frac{x}{31}\right)\right] = 500$

$x \approx 111,111$ miles

Note: Some students may recognize that the difference in costs will reach $500 again if the cars last for 222,222 miles. In this case, the MARS would cost more.

Ordering a Cab

Name ______________________ **Date** ______________________

Sarah has to take a cab to school each day. Two companies are available: Sunshine Cabs and Bluebird Cabs. Sarah conducts a survey over several months to compare the two cab companies. She orders each taxi 20 times and records how early or late they are when arriving to pick her up.

Sunshine Cabs		Bluebird Cabs	
3 min. 30 sec.	Early	3 min. 45 sec.	Late
45 sec.	Late	4 min. 30 sec.	Late
1 min. 30 sec.	Late	3 min.	Late
4 min. 30 sec.	Late	5 min.	Late
45 sec.	Early	2 min. 15 sec.	Late
2 min. 30 sec.	Early	2 min. 30 sec.	Late
4 min. 45 sec.	Late	1 min. 15 sec.	Late
2 min. 45 sec.	Late	45 sec.	Late
30 sec.	Late	3 min.	Late
1 min. 30 sec.	Early	30 sec.	Early
2 min. 15 sec.	Late	1 min. 30 sec.	Late
9 min. 15 sec.	Late	3 min. 30 sec.	Late
3 min. 30 sec.	Late	6 min.	Late
1 min. 15 sec.	Late	4 min. 30 sec.	Late
30 sec.	Early	5 min. 30 sec.	Late
2 min. 30 sec.	Late	2 min. 30 sec.	Late
30 sec.	Late	4 min. 15 sec.	Late
7 min. 15 sec.	Late	2 min. 45 sec.	Late
5 min. 30 sec.	Late	3 min. 45 sec.	Late
3 min.	Late	4 min. 45 sec.	Late

1. Analyze the data, and then use the data to present a reasoned case for Sunshine Cabs being the better company for Sarah to use in the future. Present your reasoning as fully and as clearly as possible.

2. Analyze the same data to present a reasoned case for Bluebird Cabs being the better company. Again, present your reasoning as fully and clearly as possible.

3. Which argument do you think is more convincing? Substantiate your reasoning with data.

Adapted from a Balanced Assessment Task
Balanced Assessment is funded by the National Science Foundation

Ordering a Cab

Primary Content Cluster: Statistics

Mathematics Opportunities: This task provides the student with the opportunity to:
- select appropriate calculations to analyze a set of data
- select appropriate graphs to represent and analyze the data
- plot data
- draw on data analysis and circumstances of the situation to make a recommendation
- make comparisons and decisions based on those comparisons
- present statistical information from two perspectives
- use data to defend an opinion

Fitting it in: The task can be used after students have completed a study in statistics in which they study measures of central tendency, and use of different graphic representations. The task itself is accessible to all students. However, the depth of analysis may be dependent upon prior knowledge of statistics.

A Solution: The solution provided is only a SAMPLE.

An Answer:

1. In the 20 times she called each company, Sarah had to spend 65 minutes waiting for Bluebird Cabs, while she only spent 49.75 minutes waiting for Sunshine Cabs. So she might want to go with Sunshine Cabs.

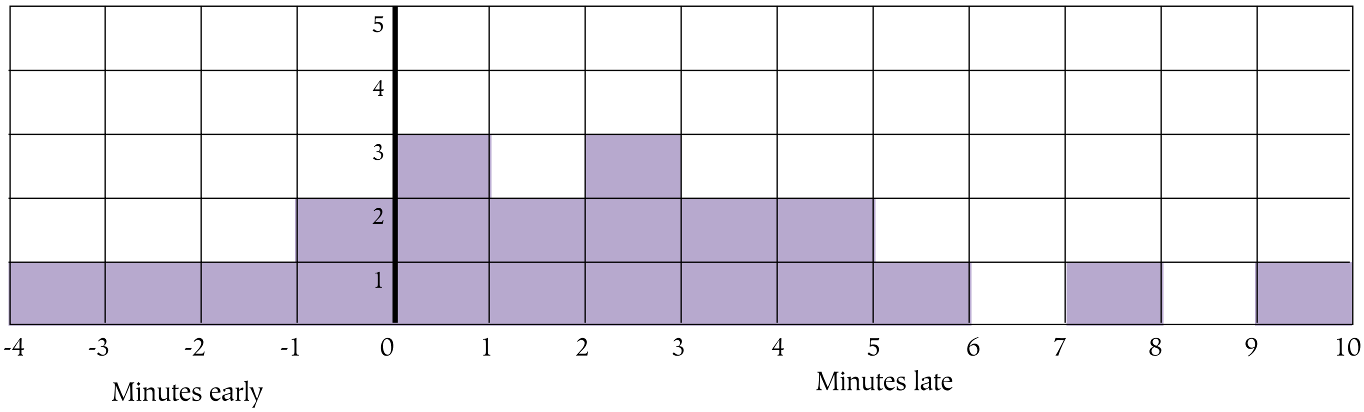
2. On the other hand, whereas Sunshine Cabs are, on average, not as late, they are less consistent in their arrival times than Bluebird Cabs. It may therefore be better to book with Bluebird Cabs, making sure that the cab is ordered for about 5 minutes before it is needed.

3. Analysis and arguments will vary.

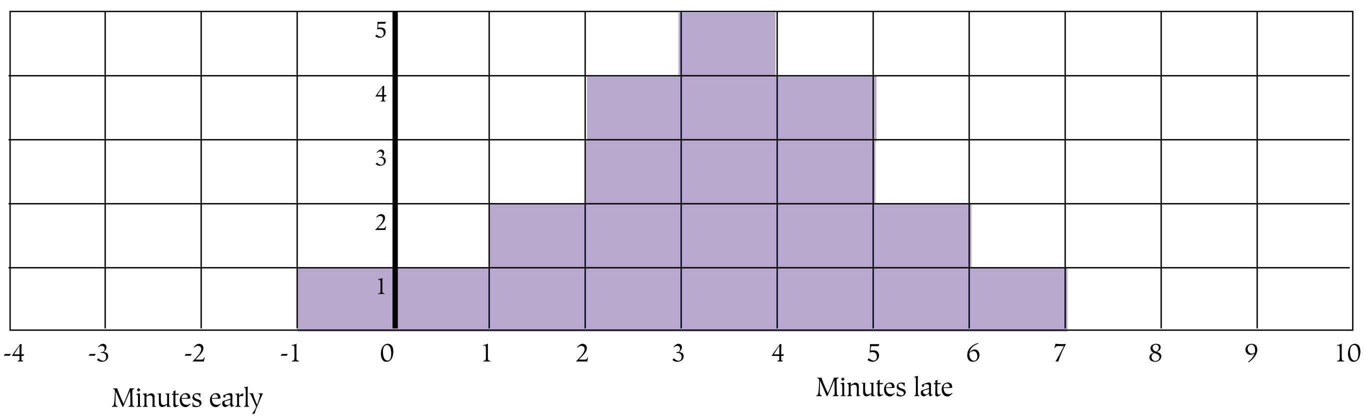
One Approach:

The data may be analyzed and graphed as follows.

Sunshine Cabs Frequency



Bluebird Cabs Frequency

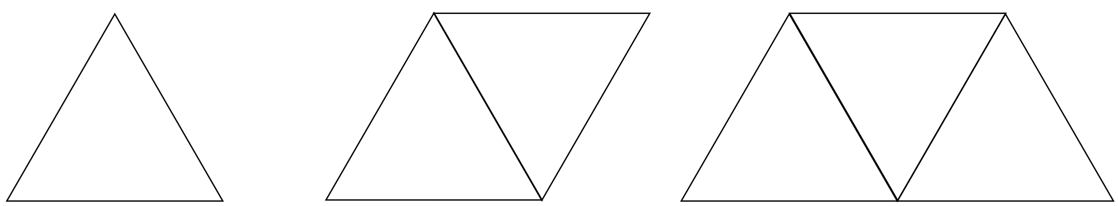


	SUNSHINE	BLUEBIRD
Mean	2 min. 3 sec.	3 min. 14 sec.
Median	1 min. 53 sec.	3 min. 15 sec.
Range	12 min. 45 sec.	6 min. 30 sec.
S D	3 min. 11 sec.	1 min. 40 sec.

Polygons in a Row

Name ________________________________ **Date** ______________________________

You are making a row of adjacent equilateral triangles, adding one at a time. Each triangle has sides with a length of 1 unit.



- What is the relationship between the number of triangles in a row and the perimeter of the new shape formed by the row of triangles?

- If the objects were squares instead of triangles, what would be the relationship between the number of squares and the perimeter?

- If the objects were equilateral hexagons instead of triangles, what would be the relationship between the number of hexagons and the perimeter?

Contributed by Wendy Ovtasky, Whitcomb Junior/Senior High School, Bethel, Vermont

Polygons in a Row

Primary Content Clusters: Algebra
Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- use some basic geometry
- generate and recognize patterns
- generalize patterns

Fitting it in: All students should be able to do this task. It is an excellent introduction to generalization of a solution. Most pre-algebra curricula provide sufficient geometry instruction for the student to understand the geometry required.

A Solution: The solution provided is only a SAMPLE.

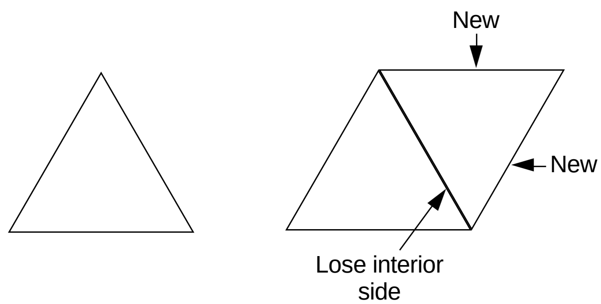
An Answer: For n triangles, the perimeter $= n + 2$
For n squares, the perimeter $= 2n + 2$
For n hexagons, the perimeter $= 4n + 2$

One Approach:

Triangles:

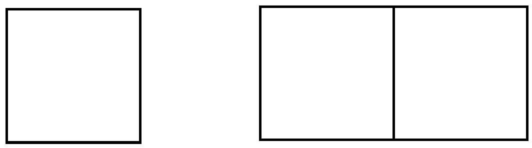
Number of Triangles	Length of Perimeter
1	3
2	4
3	5
4	6
...	...
n	$n + 2$

For each additional triangle added, we add two "new" sides to the perimeter, but we lose 1 side where the triangles abut.



Net Gain = 1

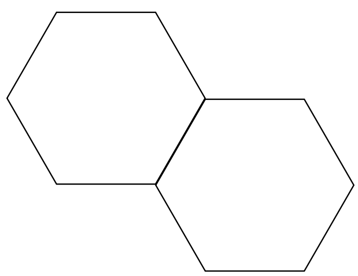
Squares:



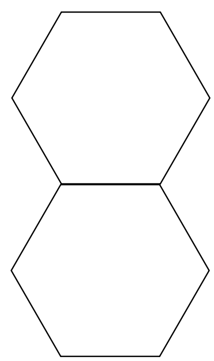
Add 3 sides
Lose 1
Net gain = 2

Number of Squares	Length of Perimeter
1	4
2	6
3	8
4	10
...	...
n	$2n+2$

Hexagons:



or



Add 5 sides
Lose 1
Net Gain = 4

Number of Hexagons	Length of Perimeter
1	6
2	10
3	14
4	18
5	22
...	...
n	$4n+2$

Note: All figures are attached in a row with only one common side. In each case, the perimeter (P) equals the number of figures in the row times the net gain, plus 2.

$$P = n(\text{Net Gain}) + 2$$

Regular Stars

Name ______________________________ **Date** ______________________________

Find the measure of the angle in each point of .....

- a regular five-pointed star
- a regular twelve-pointed star
- a regular star with any number of points

Contributed by Wendy Ovtasky, Whitcomb Junior/Senior High School, Bethel, Vermont

Regular Stars

Primary Content Clusters: Algebra
Geometry

Mathematics Opportunities: This task provides the student with the opportunity to:
- use formulas for angles of regular polygons
- use knowledge of supplementary angles and isosceles triangles
- generate and recognize a pattern
- generalize a pattern

Fitting it in: This task can be used after students have studied the angles of polygons and are familiar with the formulas involved with them. It also could be used with students who could be expected to generate those formulas on their own.

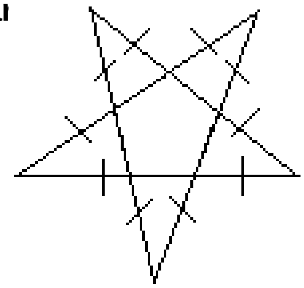
A Solution: The solution presented is only a SAMPLE.

An Answer:

- One point of a five-pointed star $= 36°$
- One point of a twelve-pointed star $= 120°$
- One point of an n-pointed star $= \frac{[360°(n-2)]}{2} - 180°$

One Approach

Five-Pointed Star

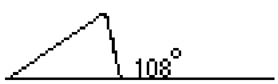


To find the measure of the interior angles of the central polygon:

$$\angle = \frac{180(n-2)}{n} = \frac{180 \cdot 3}{5} = 108°$$

(Alternatively, it can be found by dividing the central polygon into triangles and then using the angles of those trianglesto find the interior angles of the central polygon.)

The interior angles of 108 degrees are exterior to the outer isosceles triangles that make up the points.

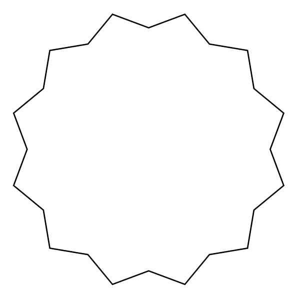


So each base $\angle = 180° - 108° = 72°$

So for the remaining angles (the points):

$\angle = 180° - 2(72°) = 36°$

Twelve-Pointed Star:



Central polygon $\angle = \dfrac{180(12-2)}{12} = 150°$

Base $\angle = 180 - 150 = 30°$

Point $\angle = 180 - 2(30) = 120°$

n-Pointed Star:

Central polygon $\angle = \dfrac{180(n-2)}{n}$ Base $\angle = 180 - \dfrac{180(n-2)}{n}$

Point $\angle = 180 - 2\left[180 - \dfrac{180(n-2)}{n}\right] = \dfrac{360(n-2)}{n} - 180$

Note: There are two possible twelve-pointed stars. Depending on which one students draw, they may reach different generalizations. Students need to provide more than the two examples (5 points and 12 points) in order to generalize to n points.

Sandbox Problem

Name ______________________________ **Date** ______________________________

Assume that you have an 8 foot by 12 foot (8' x 12') sheet of copper.

If you were to cut congruent squares out of each of the corners, you could fold up all four sides. If you then soldered the cut edges together, you would have a container which might be used as a sandbox.

How big should the cut out squares be to make the sandbox capable of holding the most sand?

Contributed by Bob Peek, Champlain Valley Union High School, Hinesburg, Vermont

Sandbox Problem

Primary Content Clusters: Functions
Algebra
Geometry
Underpinnings of Calculus

Mathematics Opportunities: This task provides the student with the opportunity to:
- apply the formula for finding the volume of a rectangular solid
- apply techniques for finding and classifying extrema
- use a graphing calculator (program) to solve a problem

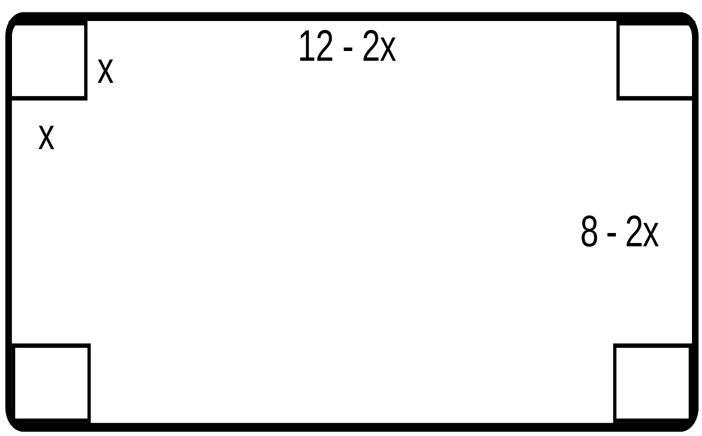
Fitting it in: This task is accessible to all students. It can be used as a post assessment in determining the volume of a rectangular prism in a problem solving situation, or assessing the understanding of the maximum value of a graphed function.

A Solution: The solution provided is only a SAMPLE.

An Answer:

The squares should be cut with approximately 1.57 feet per side.

One Approach:



The volume of the box will be:

$y = (12 - 2x)(8 - 2x)x$

$y = 4x^3 - 40x^2 + 96x$

One approach is to find y' and find when it is zero, being careful to not pick a point where x > 4.

$$y' = 12x^2 - 80x + 96$$

Setting y' to 0 yields a quadratic equation which does not factor.

$12x^2 - 80x + 96 = 0$ using the quadratic formula yields:

$$x = \frac{10 \pm 2\sqrt{7}}{3}$$

Using the + yields an impossible value for x, where the folding point is beyond $x = 4$.

So the precise solution is:

$$x = \frac{10 - 2\sqrt{7}}{3}$$

This is approximately: 1.57 ft.
Resulting volume is approximately: 67.60 cu. ft.

The approximate answer could also be found by graphing the function on a graphing calculator or with a computer, using trace-and-zoom to zero in on the high spot, or by using a spreadsheet and narrowing the range to short increments in x.

Skeleton Tower

Name ______________________________ **Date** ______________________________

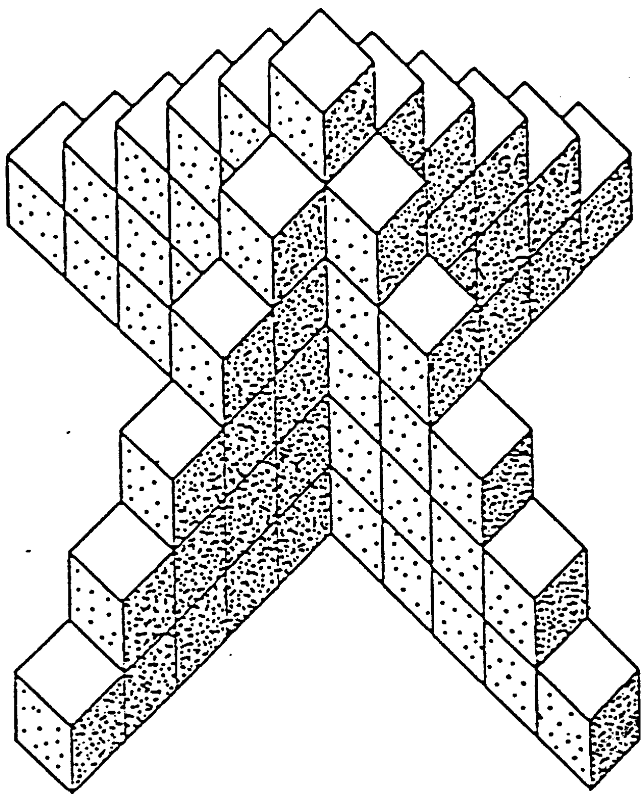
A tower is formed by stacking blocks as shown in the picture below.

Part 1:

Find out how many blocks are needed to build a tower 10 rows high.

Part 2:

Find out how many blocks are needed to build a tower n rows high.



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Skeleton Tower

Primary Content Clusters: Algebra
Discrete Mathematics

Mathematics Opportunities: This task provides the student with the opportunity to:
- organize data in a number of different ways
- work with a figure that is partially concealed
- generate and recognize a pattern
- generalize a pattern

Fitting it in: This problem is accessible for all students. A combined geometric/algebraic solution is possible; some students use the sum of an arithmetic series. Note: This task is used in many eighth grade classrooms. Ask the eighth grade teacher(s) in your district if students in your school have already solved this problem.

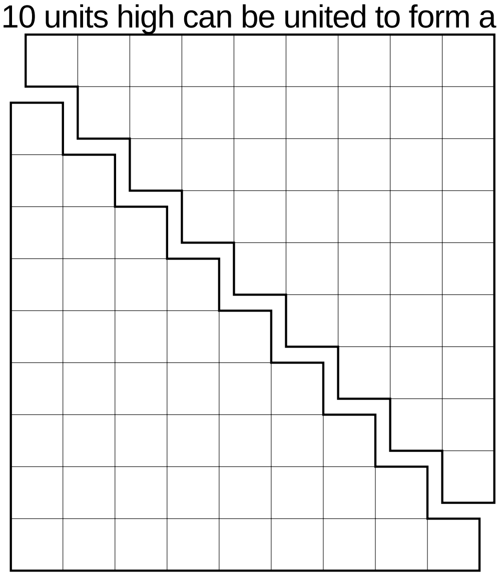
A Solution: The solution provided is only a SAMPLE.

An Answer:

Part 1: A tower 10 blocks high uses 190 cubes.
Part 2: A tower n blocks high uses $2n^2 - n$ cubes.

One Approach:

Part 1: Two “arms” of the tower 10 units high can be united to form a rectangle with dimensions 10 by 9 for an area of 90 square units.



All four arms, therefore, take $2 \cdot 90$ blocks. Adding in the center column of 10 blocks gives a total of $2(10 \cdot 9) + 10$. A tower 10 blocks high uses 190 blocks.

Part 2: In the general case, the “arms” unite to form a rectangle with dimensions n by $(n-1)$. The center column has n blocks. Therefore, the total number of blocks used is:

$2[n(n-1)]+n=2n^2-n$

Who's Who at the Movies?

Name ______________________ **Date** ______________________

Amy sells tickets at the All-Star Cinema. On Friday night, *Home Alone 34* was playing. Adults, senior citizens and children bought tickets at the following rates:

Adults	$5.00
Senior citizens	$1.00
Children	$ .10

Amy sold 100 tickets worth $100. How many adults, senior citizens and children purchased tickets?

Source: Unknown

Who's Who at the Movies?

Primary Content Cluster: Algebra

Mathematics Opportunities: This task provides the student with the opportunity to:
- work with linear equations
- work with simultaneous equations
- work with a problem which does **not** have a unique solution
- choose from a variety of techniques to solve the problem

Fitting it in: All students can do this problem by systematic trial and error. However, algebraic techniques can be used by students who know how to solve systems of linear equations. It is a good example of a system with multiple solutions.

A Solution: The solution provided is only a SAMPLE.

An Answer:

9 adults, 51 seniors, 40 children

Let $A =$ # of Adults

Let $S =$ # of Seniors

Let $C =$ # of Children

A, S, C must all be integers > 0

I $A + S + C = 100$

II $5A + S + .1C = 100$

II - I $\Rightarrow 4A - .9C = 0$

$\Rightarrow 40A = 9C$

A and C are integers $\Rightarrow$

$A = 9 \quad C = 40$

$\Rightarrow S = 51$

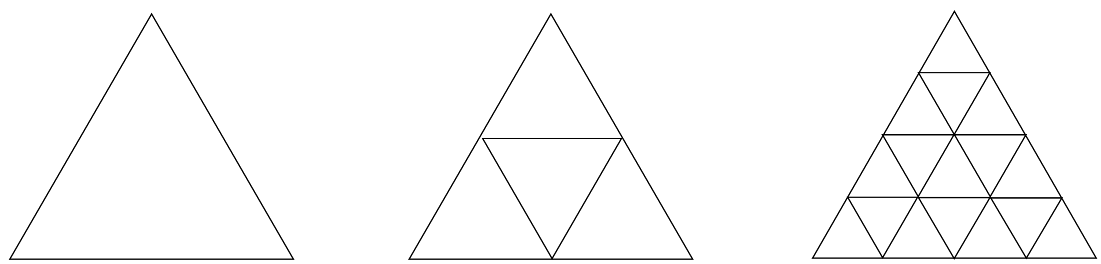
Note : 9 and 40 are relatively prime.

One Approach:

Wire Triangle

Name **Date**

An equilateral triangle of side 1 foot is constructed from wire. The midpoints of the sides of the triangle are connected with pieces of wire to form 4 equilateral triangles of side 1/2 foot. The midpoints of the sides of each of these 4 triangles are connected by wires to form 16 equilateral triangles of side 1/4 foot. This process is continued until there are 1,024 congruent equilateral triangles. What is the total length



of the wire used?

Appeared in University of Vermont High School Mathematics Contest.

Wire Triangle

Primary Content Clusters: Algebra

Mathematics Opportunities: This task provides the student with the opportunity to:
- organize data
- generate and recognize patterns
- generalize a pattern

Fitting it in: This problem is appropriate for all students. It can fit at any time in the curriculum. It may be best used after students have discussed midpoints, congruence, etc. This will allow students to use the vocabulary in a new problem-solving situation. For a student solution, refer to the Benchmarks.

A Solution: The solution provided is only a SAMPLE.

An Answer:

49.5 feet of wire

One Approach:

The table below was generated by modeling the first three cases and then extending the table.

n	# of small triangles	length of one side in ft.	# of added sides	added length in ft.	Total Length
1	1	1			6/2
2	4	1/2	3	3/2	9/2
3	16	1/4	12	6/2	15/2
4	64	1/8	48	12/2	27/2
5	256	1/16	192	24/2	51/2
6	1,024	1/32	768	48/2	99/2
...	...	...	...	...	...
Although it is not asked for, a generalization may be reached for any of these columns:					
n	4^{n-1}	$\frac{1}{2^{n-1}}$	$3(4^{n-2})$	$\frac{3(2^{n-2})}{2}$ or $3(2^{n-3})$	$\frac{3(2^{n-1}+1)}{2}$

There are 1,024 congruent triangles on the 6th iteration.
So, the total length of wire used will be 49.5 feet.

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